

A Capacitor Paradox

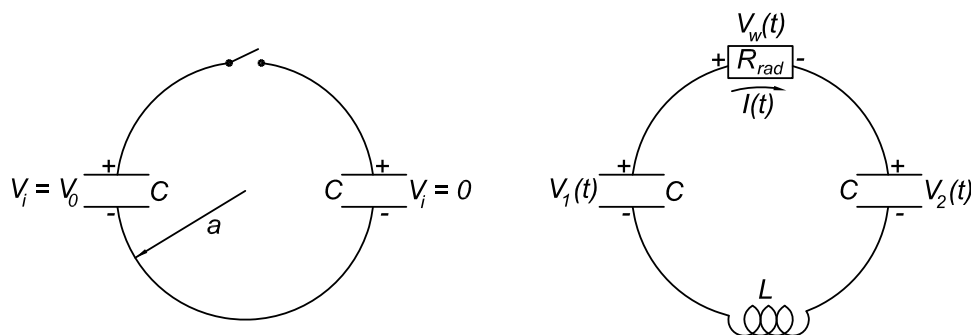
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1 Problem

Two capacitors of equal capacitance C are connected in parallel by zero-resistance wires and a switch, as shown in the lefthand figure below. Initially the switch is open, one capacitor is charged to voltage V_0 and the other is uncharged. At time $t = 0$ the switch is closed. If there were no damping mechanism, the circuit would then oscillate forever, at a frequency dependent on the self inductance L and the capacitance C . However, even in a circuit with zero Ohmic resistance, damping occurs due to the radiation of the oscillating charges, and eventually a static charge distribution results.



1. How much energy is stored in the system before and (a long time) after the switch is closed? The paradox is that these two energies are not equal.
2. Verify that the “missing” stored energy has been radiated away by the transient current after the switch was closed, supposing that the Ohmic resistance of all circuit components is negligible. (Recall that in Poynting’s view, the energy that is transferred from one capacitor to the other passes through the intervening space, not down the connecting wires.)

For this, you may assume that the wires form a circle of radius a and that the conduction currents in the capacitor plates may be neglected. In this approximation the current in the wires is spatially uniform. You may neglect the charge accumulation in the wires compared to that on the capacitor plates, in which case the total electric dipole moment of the system is constant and magnetic dipole radiation dominates. Also ignore the self inductance of the circuit.

The “radiation resistance” of this circuit causes a voltage drop V_w to occur along the nominally zero-resistance wires that can be identified as

$$V_w(t) = \frac{P_{\text{rad}}(t)}{I(t)}, \quad (1)$$

where P_{rad} is the radiated power and $I(t)$ is the current in the wire. That is, the radiating wires act like a third circuit element that can be combined with the two capacitors in an analysis of the voltage and current in the circuit, as sketched in the equivalent circuit above.

From this circuit analysis deduce a (nonlinear) differential equation for the difference in voltages on the two capacitors, $V_{12} = V_1 - V_2$, which has an exponential solution. Using this solution, verify that the total radiated energy is equal to the loss of energy stored in the capacitors after the switch is closed. Also, give the magnitude of the radiation resistance (in Ohms).

3. Extend the analysis to include self inductance.

2 Solution

This problem is adapted from [1]. Two earlier discussions that neglected radiation are [2, 3].

1. The energy stored in a capacitor of capacitance C that is charged to voltage V is $U = CV^2/2$. Hence the initial stored energy of the system is

$$U_i = \frac{CV_0^2}{2}. \quad (2)$$

A long time after the switch has been closed the initial charge is equally distributed between the two capacitors, each of which is now at voltage $V_0/2$. Hence, the final stored energy of the system is

$$U_f = 2 \frac{C(V_0/2)^2}{2} = \frac{CV_0^2}{4} = \frac{U_i}{2}. \quad (3)$$

2. To calculate the radiated power we note that the stated approximations are such that electric multipole radiation by the circuit may be neglected, and magnetic dipole radiation dominates. The magnetic moment m of the circuit is (in Gaussian units)

$$m(t) = \frac{\pi a^2 I(t)}{c}, \quad (4)$$

where c is the speed of light. According to the Larmor formula, the radiated power is

$$P_{\text{rad}} = \frac{2\ddot{m}^2}{3c^3} = \frac{2\pi^2 a^4 \ddot{I}^2}{3c^5}. \quad (5)$$

As suggested, we introduce the voltage drop V_w along the wires due to the radiation resistance via

$$V_w = \frac{P_{\text{rad}}}{I} = \frac{2\pi^2 a^4 \ddot{I}^2}{3c^5 I}. \quad (6)$$

The current I is also related to the charge and voltage on the capacitors by

$$\dot{V}_1 = \frac{\dot{Q}_1}{C} = -\frac{I}{C}, \quad \text{and} \quad \dot{V}_2 = \frac{\dot{Q}_2}{C} = \frac{I}{C} \quad (7)$$

An additional relational between the currents and voltages is obtained from Kirchhoff's circuit law,

$$V_1 - V_2 + L\dot{I} - V_w = 0, \quad (8)$$

including the self-inductance term for later consideration.

Our strategy now is to use $V_{12} = V_1 - V_2$ as the independent variable, whose initial value is V_0 . From eq. (7) we have

$$I = -\frac{C\dot{V}_{12}}{2}, \quad \text{and hence,} \quad \dot{I} = -\frac{C\ddot{V}_{12}}{2}, \quad \ddot{I} = -\frac{C\dddot{V}_{12}}{2}. \quad (9)$$

Using eqs. (6) and (9) in eq. (8) we find the differential equation for V_{12} to be

$$\ddot{V}_{12}^2 + \frac{3c^5 L}{2\pi a^4} \dot{V}_{12} \ddot{V}_{12} + \frac{3c^5}{\pi a^4 C} V_{12} \dot{V}_{12} = 0. \quad (10)$$

We try a solution of the form

$$V_{12}(t > 0) = V_0 e^{-t/\tau}, \quad (11)$$

which satisfies eq. (10) provided

$$\frac{1}{\tau^6} = \frac{3c^5 L}{2\pi a^4 \tau^3} + \frac{3c^5}{\pi a^4 C \tau}. \quad (12)$$

We first suppose that we may set the self inductance L to zero. Of the six possible solutions for τ in this case, we choose the only one that is real and finite,

$$\tau = \left(\frac{\pi a^4 C}{3c^5} \right)^{1/5}. \quad (13)$$

The capacitance C in eq. (13) is representative of the total capacitance of the system, which can never be much smaller than length a , since the capacitance of the wires alone has roughly this value. Hence, the time constant of the discharge of the capacitors obeys $\tau \geq a/c$. That is, the discharge time is longer than the transit time of light across the circuit.

The total radiated power can now be calculated, combining eqs. (5), (9), (11) and (13),

$$U_{\text{rad}} = \int_0^\infty P_{\text{rad}} dt = \int_0^\infty \frac{2\pi^2 a^4 C^2 V_0^2 e^{-2t/\tau}}{3c^5 4\tau^6} dt = \frac{\pi^2 a^4 C^2 V_0^2}{3c^5 4\tau^5} = \frac{C V_0^2}{4} = U_i - U_f, \quad (14)$$

and is indeed equal to the “missing” stored energy.

Finally, we identify the radiation resistance by combining eqs. (8), (9) and (11) (ignoring the self inductance) in the form

$$V_w = \frac{2\tau}{C} I \equiv I R_{\text{rad}}. \quad (15)$$

Thus,

$$R_{\text{rad}} = \frac{2\tau}{C} = \frac{1}{c} \left(\frac{\pi}{3} \right)^{1/5} \left(\frac{a}{C} \right)^{4/5} \lesssim \frac{1}{c} = 30\Omega. \quad (16)$$

Since this value is large compared to the resistance of typical lead wires in a circuit, our approximation of zero-resistance wires is a good one (unless $C \gg a$).

3. In extending the analysis to include self induction, it may be useful to first recall the behavior of the circuit if radiation is neglected. Then, the circuit equation (10) reduces to the form

$$\ddot{V}_{12} + \frac{2}{LC}V_{12} = 0, \quad (17)$$

which has the oscillatory solution (for the stated initial conditions)

$$Q_1 = \frac{CV_0}{2}(1 + \cos \omega t), \quad (18)$$

$$Q_2 = \frac{CV_0}{2}(1 - \cos \omega t), \quad (19)$$

$$I = -\dot{Q}_1 = \dot{Q}_2 = \frac{CV_0\omega}{2} \sin \omega t, \quad (20)$$

$$(21)$$

where the oscillation frequency is

$$\omega = \sqrt{\frac{2}{LC}}. \quad (22)$$

The stored energy is

$$U = \frac{1}{2}LI^2 + \frac{Q_1^2 + Q_2^2}{2C} = \frac{CV_0^2}{2} = U_0, \quad (23)$$

which is constant in time at its initial value, since there is no dissipative mechanism by assumption.

The self inductance of a circular loop (a torus) of wire of thickness $2b$ (minor radius b and major radius $a \gg b$) is given by

$$L = \frac{4\pi a}{c^2} \left(\ln \frac{8a}{b} - \frac{7}{4} \right). \quad (24)$$

For $a/b \approx 100$, the self inductance is $L \approx 60a/c^2$. The LC oscillation of the circuit has time constant $\approx \sqrt{LC} \approx \sqrt{60aC/c^2} \approx 8a/c \approx 8\tau_{\text{rad}}$, where we again suppose that $C \approx a$. Thus, for reasonable circuit parameters, the radiation damping time is of the same order of magnitude as the LC oscillation period. In practice, we can have either over- or underdamped oscillations, depending on the parameter values.

In the case of overdamped “oscillations”, the self inductance is largely ignorable and the analysis given in item 2 is still valid. For weakly damped oscillations, the second term on the righthand side of eq. (12) is small compared to the first, so an iterative solution for $1/\tau$ follows readily. But as the details are not particularly illuminating we leave them to readers more motivated by practice than by principle.

References

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