"Crossed-Field" and "EH" Antennas Including Radiation from the Feed Lines and Reflection from the Earth's Surface

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1 Problem

An ongoing challenge in electrical engineering is the design of antennas whose size is small compared to the broadcast wavelength λ. One difficulty is that the radiation resistance of a small antenna is small compared to that of the typical transmission lines that feed the antenna,[1] so that much of the power in the feed line is reflected off the antenna rather than radiated.

The radiation resistance of an antenna that emits dipole radiation is proportional to the square of the peak (electric or magnetic) dipole moment of the antenna. This dipole moment is roughly the product of the peak charge times the length of the antenna in the case of a linear (electric) antenna, and is the product of the peak current times the area of the antenna in the case of a loop (magnetic) antenna. Hence, it is hard to increase the radiation resistance of small linear or loop antennas by altering their shapes.[2]

One suggestion for a small antenna is the so-called "crossed-field" antenna [2]. Its proponents are not very explicit as to the design of this antenna, so this problem is based on a conjecture as to its motivation.[3]

It is well known that in the far zone of a dipole antenna the electric and magnetic fields have equal magnitudes (in Gaussian units), and their directions are at right angles to each other and to the direction of propagation of the radiation. Furthermore, the far zone electric and magnetic fields are in phase. The argument is, I believe, that it is desirable if these conditions could also be met in the near zone of the antenna.

The proponents appear to argue that in the near zone the magnetic field $\mathbf{B}$ is in phase with the current in a simple, small antenna, while the electric field $\mathbf{E}$ is in phase with the charge, but the charge and current have a 90° phase difference. Hence, they imply, the electric and magnetic fields are 90° out of phase in the near zone, so that the radiation (which is proportional to $\mathbf{E} \times \mathbf{B}$) is weak.

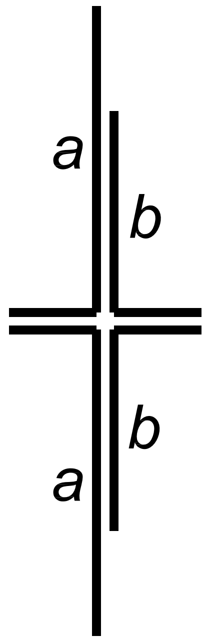
The concept of the "crossed-field" antenna seems to be based on the use of two small antennas driven 90° out of phase. The expectation is that the electric field of one of the antennas will combine with the magnetic field of the other to produce radiation that is much more powerful than that from either of the two antennas separately.

[1] A center-fed linear dipole antenna of total length $l \ll \lambda$ has radiation resistance $R_{\rm linear} = (l/\lambda)^2\, 197\,\Omega$, while a circular loop antenna of diameter $d \ll \lambda$ has $R_{\rm loop} = (d/\lambda)^4\, 1948\,\Omega$. For example, if $l = d = 0.1\lambda$ then $R_{\rm linear} = 2\,\Omega$ and $R_{\rm loop} = 0.2\,\Omega$.

[2] That there is little advantage to so-called small fractal antennas is explored in [1].

[3] A variant based on combining a small electric dipole antenna with a small magnetic dipole (loop) antenna [3] is discussed in Appendix D.

If suffices to consider two small linear dipole antennas, say of lengths $2a \ll \lambda$ and $2b \ll \lambda$, as shown in the figure below. Discuss the dependence of the total power radiated by the two antennas as a function of a, b, and the drive currents $I_a e^{-i\omega t}$ in antenna a and $I_b e^{-i(\omega t+\phi)}$ in antenna b.



A variant of the "crossed-field" antenna is the so-called "EH" antenna [4, 5],[4] which is a short, linear dipole antenna in which the currents in the two arms are driven 90° out of phase.[5] Discuss the power radiated by this antenna.

An important aspect of practical antennas is the behavior of the feed line between the rf power source and the antenna. Ideally this is a two-wire transmission line, such as a coaxial cable, that carries equal and opposite currents on the two wires. Then the radiation from the currents in the two wires cancels and the feed line can be ignored when discussing radiation by the antenna itself.

The pair of dipole antennas that comprise the "crossed-field" antenna can be operated with their drive currents 90° out of phase simply and precisely by connecting them to a pair of coaxial cables from a single rf power source such that one cable is $\lambda/4$ longer than the other. Then each coaxial cable operates as an ideal, nonradiating transmission line and these lines can be neglected in an analysis of the radiation.[6]

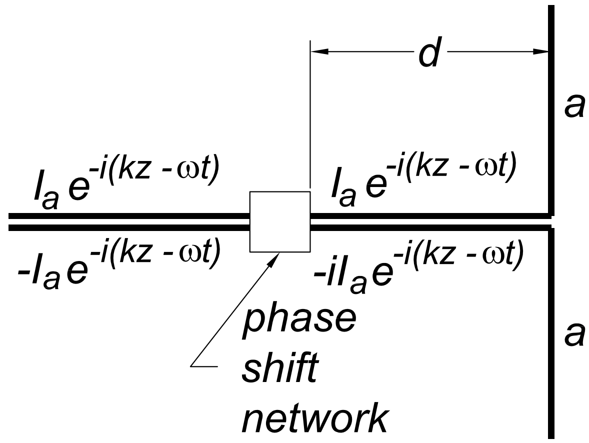
However, an "EH" antenna cannot be driven by a single, nonradiating transmission line in which the currents on its two conductors are opposite rather than 90° out of phase.

[4] Reference [4] also discusses the so-called "H_z" antenna which consists of a pair of small, loop antennas placed side by side and driven 90° out of phase. The behavior of this antenna pair is essentially the same as that of a "crossed-field" antenna and need not be discussed separately.

[5] In one definition of the EH antenna [6], it is recommended that the antenna include "tuning" coils such that when the system is considered as two-terminal device, the applied voltage would be exactly 90° out of phase with the current. Since this implies that the system would draw zero power, you need not consider this suggestion further.

[6] If "crossed-field" antennas had been driven in this manner, the controversy as to their performance might have been settled long ago.

As shown in the figure below, suppose that a small phase shift network is installed in the transmission line at a distance $d \approx \lambda/4$ from the feed points of an "EH" antenna, such that over the length d the current in one conductor is $I_a e^{-i\omega t}$ and that in the other conductor is $-iI_a e^{-i\omega t}$ (where currents in the figure are positive when flowing to the right and when flowing upwards).



Deduce the contribution to the radiation from the currents in this feed line.

Another aspect of practical reality is that antennas are mounted close to the Earth's surface, which acts like a perfectly conducting ground plane to a reasonable approximation. Discuss the effect of this ground plane on the behavior of the "crossed-field" and "EH" antennas.

2 Solution

2.1 The "Crossed-Field" Antenna

We recall that the time-averaged power P radiated by an antenna system with total electric dipole moment $\mathbf{p} = \mathbf{p}_0 e^{-i\omega t}$ is

$$P = \frac{|\ddot{\mathbf{p}}|^2}{3c^3} = \frac{\omega^4 |\mathbf{p}_0|^2}{3c^3}. \tag{1}$$

For completeness, we deduce the dipole moment of a center-fed linear antenna of length $2a$. We take the conductors to be along the z-axis, with the feed point at the origin. The current at the feed point is $I_a e^{-i\omega t}$, but it must fall to zero at the tips of the antenna $z = \pm a$. When $a \ll \lambda$ the current distribution can only have a linear dependence on z, so its form must be[7]

$$I(z,t) = I_a e^{-i\omega t} \left(1 - \frac{|z|}{a}\right). \tag{2}$$

[7]Strictly speaking, only the current that is in phase with the drive current must have the form (2). There actually exists a small current that is 90° out of phase with the drive current, and which vanishes at $z = 0$ as well as $z = \pm a$. This current is needed to provide some additional electric field in the near zone such that the tangential component of the total electric field vanishes along the (good) conductors. However, this current does not affect the radiated power, and may be neglected in the present discussion.

The charge distribution $\varrho(z,t) = \rho(z)e^{-i\omega t}$ along the antenna can be deduced from the current distribution (2) using the equation of continuity (*i.e.*, charge conservation), which has the form $\partial I/\partial z = -\partial\varrho/\partial t = i\omega\rho(z)e^{-i\omega t}$. Thus,

$$\rho(z) = \pm\frac{iI_a}{\omega a}. \tag{3}$$

The electric dipole moment p_a is therefore

$$p_a(t) = \int_{-a}^{a} z\rho(z)e^{-i\omega t}\, dz = \frac{iI_a a}{\omega}e^{-i\omega t}. \tag{4}$$

Similarly, the electric dipole moment of antenna b, whose current is $I_b e^{-i(\omega t+\phi)}$, is

$$p_b(t) = \frac{iI_b b}{\omega}e^{-i(\omega t+\phi)}. \tag{5}$$

The total electric dipole moment for the two antennas of the present example is

$$p(t) = i\frac{I_a a + I_b b e^{i\phi}}{\omega}e^{-i\omega t} \equiv p_0 e^{-i\omega t}. \tag{6}$$

The total (time-averaged) power radiated by the "crossed-field" antenna is, using eq. (1),

$$P = \frac{\omega^4\,|\mathbf{p}_0|^2}{3c^3} = \frac{\omega^2}{3c^3}(I_a^2a^2 + I_b^2b^2 + 2I_a a I_b b\cos\phi) = P_a + P_b + 2\sqrt{P_a P_b}\cos\phi, \tag{7}$$

where P_a and P_b are the powers that would be radiated by each antenna in the absence of the other.

2.2 The "EH" Antenna

We take the total length of the antenna to be $2a$ along the z-axis. The drive current is assumed to have the form

$$I(z,t) = I_a e^{-i\omega t}\left(1 - \frac{|z|}{a}\right)\begin{cases} 1 & (0 < z < a), \\ i & (-a < z < 0), \end{cases} \tag{8}$$

which incorporates a 90° phase difference between the currents in the two arms of the antenna. Comparing with eq. (3), we see that the distribution of charge along the antenna has the form

$$\rho(z) = \frac{I_a}{\omega a}\begin{cases} i & (0 < z < a), \\ 1 & (-a < z < 0). \end{cases} \tag{9}$$

The electric dipole moment p_a of the antenna is therefore

$$p_a(t) = \int_{-a}^{a} z\rho(z)e^{-i\omega t}\, dz = \frac{I_a a}{2\omega}(1+i)e^{-i\omega t} \equiv p_0 e^{-i\omega t} \tag{10}$$

The time-averaged power radiated by the "EH" antenna is, using eq. (1),

$$P = \frac{\omega^4\,|\mathbf{p}_0|^2}{3c^3} = \frac{\omega^2 a^2 I_a^2}{6c^3} = \frac{P_a}{2}, \tag{11}$$

where $P_a = \omega^2 a^2 I_a^2/3c^3$ is the power that would be radiated if the two arms of the antenna were driven in phase.

2.3 Radiation from the Feed Line to the "EH" Antenna

When a feed line contains "unbalanced" currents, as specified in the "EH" antenna scheme, there is radiation from the feed line as well as from the antenna proper. When an "unbalanced" feed line is long compared to the size of the antenna, the radiation from the feed line is much larger than that of the antenna proper. Here, we deduce the radiation from the feed line shown in the figure on p. 3.

We take the lower conductor of the feed line to be the inner conductor of the coaxial cable. Then, the current on the inner conductor is $-iI_a e^{i(kz-\omega t)}$, where $k = 2\pi/\lambda = \omega/c$, and the feed line is taken to lie along the z-axis.[8] Anticipating further analysis in the following section, we suppose the feed line is vertical and extends from $z = 0$ up to $z = d$.

Accompanying the current on the inner conductor is a current $iI_a e^{i(kz-\omega t)}$ on the inside of the outer conductor. These equal and opposite currents are associated with the transverse electromagnetic wave (TEM) that propagates inside the coaxial cable. These two current do not directly produce any radiation in the far zone.[9]

In addition, there is a current $I_o e^{i(kz-\omega t)}$ that flows on the outside of the outer conductor. The radiation from the feed line is due to this current.

Since the total current on the outer conductor is $I_a e^{i(kz-\omega t)}$, we have that

$$I_o e^{i(kz-\omega t)} = I_a e^{i(kz-\omega t)} - (iI_a e^{i(kz-\omega t)}) = (1-i) I_a e^{i(kz-\omega t)}. \tag{12}$$

The radiating current (12) extends over a distance d that is not small compared to a wavelength. In this case, an accurate calculation of the radiation should go beyond the dipole approximation. Fortunately, there is an "exact" prescription for the (time-averaged) angular distribution of radiation in the far zone from a specified time-harmonic current distribution [7],

$$\frac{dP}{d\Omega} = \frac{\omega^2}{8\pi c^3} \left| \hat{\mathbf{k}} \times \left[\hat{\mathbf{k}} \times \int \mathbf{J}(\mathbf{r}, t) e^{-i\mathbf{k}\cdot\mathbf{r}} \, d\text{Vol} \right] \right|^2 . \tag{13}$$

In the present problem $\mathbf{J}(\mathbf{r}, t)\, d\text{Vol} \to I_o e^{i(kz-\omega t)}\, \hat{\mathbf{z}}\, dz$. For a far-zone observer at angles (θ, ϕ) in a spherical coordinate system, the unit vector $\hat{\mathbf{k}}$ is given by

$$\hat{\mathbf{k}} = \sin\theta\cos\phi\ \hat{\mathbf{x}} + \sin\theta\sin\phi\ \hat{\mathbf{y}} + \cos\theta\ \hat{\mathbf{z}}. \tag{14}$$

Combining eqs. (12)-(14), we have

$$\frac{dP(\theta,\phi)}{d\Omega} = \frac{\omega^2 I_a^2}{4\pi c^3} \sin^2\theta \left| \int_0^d e^{ikz(1-\cos\theta)}\, dz \right|^2 = \frac{\omega^2 I_a^2 d^2}{4\pi c^3} \sin^2\theta \left[\frac{\sin\frac{kd}{2}(1-\cos\theta)}{\frac{kd}{2}(1-\cos\theta)} \right]^2 . \tag{15}$$

The factor in eq. (15) in brackets represents the departure of the radiation pattern from that of an ideal dipole, due to the finite length d of the feed line. The factor is 1 for $\theta = 0^\circ$ and 180° for any value of d. At $\theta = 90^\circ$ the (factor)^2 is 0.81 for $d = \lambda/4$ and 0.41 for $d = \lambda/2$.

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[8] The speed of light inside the coax is in general less than c, the speed of light in vacuum. We ignore this detail in the present analysis.

[9] The TEM wave inside the coaxial feed line transfers power from the rf source to the antenna, the flow of which is described by the (time-averaged) Poynting vector. Some of the power inside the coax cable is "radiated" into the far zone, and lines of the Poynting vector in the far zone are direct continuations of lines of the Poynting vector inside the cable. From the perspective of power flow, the antenna structure, along with the coaxial feed line, merely "guides" the transmission of rf power from the source into the far zone.

The radiation pattern (15) is generated by the traveling-wave current of eq. (12), and which is sometimes called the "Beverage" antenna pattern (US patent 1,381,089, June 7, 1921) [8]. Typical "Beverage" antennas are several wavelengths long to increase their directivity (along the direction of the traveling wave).

For comparison, the radiation pattern of a center-fed linear dipole antenna of total length d is

$$\frac{dP(\theta,\phi)}{d\Omega} = \frac{I^2}{2\pi c}\left[\frac{\cos(\frac{kd}{2}\cos\theta) - \cos\frac{kd}{2}}{\sin\frac{kd}{2}\sin\theta}\right]^2 . \tag{16}$$

At $\theta = 90^\circ$ the square of the factor in brackets in eq. (16) is 0.09 for $d = \lambda/4$ and 1.0 for $d = \lambda/2$. For $d < \lambda/2$, the radiation pattern of the "EH" feed line is more strongly peaked in the equatorial plane than is the pattern of a center-fed linear antenna.

We can integrate eq. (15) over angles to obtain the total (time-averaged) power radiated by the "EH" feed line, but there is no simple analytic result. As an approximation, we ignore the factor in brackets, which gives[10]

$$P = \int \frac{dP(\theta,\phi)}{d\Omega}\, d\Omega \approx \frac{2\omega^2 d^2 I_a^2}{3c^3} . \tag{17}$$

When $d \gg a$ this is large compared to the power $P_a = \omega^2 a^2 I_a^2/6c^3$ radiated by the arms of the antenna, as found in eq. (11). Hence, with the phase shift network located at a distance $d \gg a$ from the antenna, as recommended in [5], most of the radiation of an "EH" antenna actually comes from the feed line between the phase shift network and the antenna, rather than from the antenna itself.[11]

2.4 Effects of the "Ground" Plane

The discussion thus far has presumed that the antennas are located in free space, far from any other conductors. But in many cases antennas are arrayed a small distance above the surface of the Earth, which acts like an ideal ground plane to some approximation.

Then, all charges in the system can be thought of as having "image" partners of the opposite sign located at distances underground equal to the heights of the actual charges above ground. The oscillating "image" charges create radiation (actually due to currents on the surface of the Earth), that interferes with the radiation from the nominal antenna.

One option is to build only one, vertical arm of an antenna, and connect the "return" conductor of the feed line to "ground" directly below that single arm. Then the "image" of the single arm acts like the second arm of a dipole antenna, whose behavior is like that of the antennas modeled here to the extent that the Earth is a perfect conductor. The "crossed-field" antenna of [2] is based on a single physical arm plus "image" arm. Hence, its behavior is reasonably well modeled by the analysis of sec. 2.1.

The "EH" antenna of [5] is stated to be mounted with its center at distance $d = \lambda/8$-$\lambda/4$ above the Earth's surface. The dipole moment of the charges on the conductor of the

[10] Equation (17) is obtained in the approximation that the length d of the feed line is small compared to the wavelength λ. In this approximation the current $I_o e^{-i\omega t}$ has no spatial dependence. Then, no charge accumulates along the feed line itself. Rather, charges $\pm Q = \pm i I_o/\omega$ accumulate at the two ends of the feed line. The electric dipole moment of these charges is Qd, from which eq. (17) could also be obtained.

[11] An empirical study of radiation from the feed line of an "EH" antenna is given in [9].

antenna has an "image" dipole of the opposite sign located at distance d underground. The radiation patterns of the antenna dipole and its "image" interfere constructively if $d = \lambda/4$ [10], but the radiation from the antenna dipole is negligible compared to that from the feed line if $d \gg a$.

Presumably the phase shift network is on the ground at the base of the antenna tower, such that the feed line runs upwards from $z = 0$ to $z = d$. The current $I_o e^{i(kz-\omega t)}$ on the outside of the feed line creates an "image" current of the same form. That is, the "image" of an upward moving positive charge is a downward moving negative charges, and both of these motions corresponds to upward currents. The (time-averaged) radiation pattern of the feed line plus its image can be calculated as in eq. (15),

$$\frac{dP(\theta,\phi)}{d\Omega} = \frac{\omega^2 I_a^2}{4\pi c^3} \sin^2\theta \left| \int_{-d}^{d} e^{ikz(1-\cos\theta)}\, dz \right|^2 = \frac{\omega^2 I_a^2 d^2}{\pi c^3} \sin^2\theta \left[\frac{\sin(kd(1-\cos\theta))}{kd(1-\cos\theta)} \right]^2 . \qquad (18)$$

The total (time-averaged) radiated power, neglecting the factor in brackets, is approximately

$$P \approx \frac{8\omega^2 d^2 I_a^2}{3c^3} \equiv \frac{I_a^2}{2} R_{\rm rad}\,, \qquad (19)$$

where the radiation resistance is

$$R_{\rm rad} = \frac{16\omega^2 d^2}{3c^3} = \frac{64\pi^2 d^2}{3\lambda^2 c} \approx 6300 \frac{d^2}{\lambda^2}\,\Omega, \qquad (20)$$

noting that $1/c = 30\ \Omega$. For $d = \lambda/8$ the radiation resistance is about 100 Ω.

The length a of the conductors of the "EH" antenna plays no role in this result, and these conductors could be shortened to zero length with no change in the performance of the system, whose radiation is due to the feed line, and the antenna itself.

3 Comments

Equation (7) tells us that a "crossed-field" antenna pair would actually work better if both antennas were driven in phase than if they are driven 90° out of phase ($\phi = 90°$) as recommended by its proponents. Similarly, eq. (11) tells us that an "EH" antenna would work better if both arms of the antenna were driven in phase and if the phase shift network were located at the feed point of the antenna.

Sections. 2.3 and 2.4 show that when the phase shift network of an "EH" antenna is located on the ground but the antenna is mounted at height d, radiation from the feed line becomes more important than that from the nominal antenna, and the effective length of the antenna is $2d$ rather than the length a of the nominal arms of the antenna. In this case, the "EH" antenna is better described as a traveling-wave antenna of the type introduced by Beverage in 1921 [8].

A single linear dipole antenna actually satisfies the *desiridata* of a "crossed-field" antenna. From Appendix A, which is based on work by Kliatzkin in 1927 [11], we see that the radiation fields can be identified in the near zone of a simple linear antenna, where the magnitudes of the electric and magnetic radiation fields are essentially equal, and they are in phase. Adding a second antenna and varying its phase difference with respect to the first does NOT improve the quality of the system as an antenna.[12] Changing the shape of the antenna helps only to the extent that it increases the electric dipole moment while keeping the overall antenna size fixed.

In Appendix B we discuss how the electric and magnetic fields of a short, linear antennas would by very different if the "displacement current" could be neglected. It is sometimes said that the main effect of "displacement current" is to produce radiation, which is a small effect in the near zone of a system. But to the contrary, the "displacement current" has a large effect on the nonradiation part of the fields in the near zone, especially in situations where radiation is produced, so that it is a very poor strategy to neglect the "displacement current" in an attempt to gain a quick understanding of the near fields. In particular, the electric and magnetic fields have in-phase components in the near zone (in examples like the present) when the "displacement current" is included, while they would be 90° out of phase if the "displacement current" could be neglected.

Appendix C presents general expressions for the electric and magnetic fields of a specified set of charge and current distributions. These forms give an additional perspective as to how the inclusion of the "displacement current" in Maxwell's equations leads to expressions that can be thought of as retarded static fields, plus radiation fields, plus a third term in the electric field that is significant close to the source. The radiation fields have the character of "crossed fields" both near to and far from the source.

A Appendices

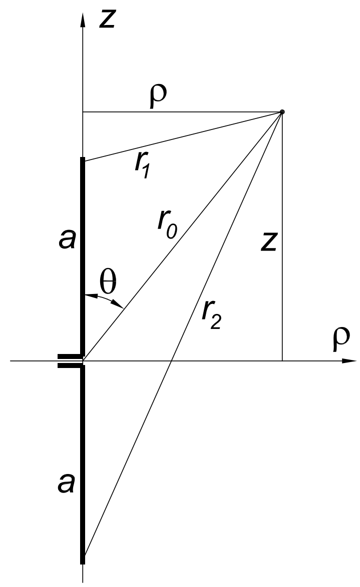
A.1 Appendix A: Near Fields of a Linear Dipole Antenna

This Appendix reproduces results from sec. 2.3 of [12] as to the near fields of a linear dipole antenna of length $2a$, as shown in the figure below, with an assumed current distribution

$$I(z,t) = I_0 \sin[k(a - |z|)] \cos \omega t, \tag{21}$$

which is not normalized: $I(z = 0) = I_0 \sin ka \cos \omega t$.

[12]There is some technical interest in an antenna composed of two linear antenna at right angles to one another, and driven 90° out of phase. This so-called "turnstile" antenna produces circularly polarized radiation. See, for example, sec. 3 of [13].



The electric and magnetic fields can be calculated from the retarded vector potential, which has only a z-component in this example,

$$A_z(\mathbf{x},t) = \frac{1}{c}\int_{-a}^{a} dz' \; \frac{I(z', t' = t - R/c)}{R} = \frac{I_0 e^{-i\omega t}}{c}\int_{-a}^{a} dz' \; \sin\sin[k(a - |z'|)]\frac{e^{ikR}}{R}, \tag{22}$$

where $k = 2\pi/\lambda = \omega/c$ and $R = |\mathbf{x} - \mathbf{x}'|$. Then, the fields $\mathbf{E}$ and $\mathbf{B}$ are related by[13]

$$\mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}, \qquad \text{and} \qquad \left[\frac{i}{kc}\frac{\partial\mathbf{E}}{\partial t} =\right]\mathbf{E} = \frac{i}{k}\boldsymbol{\nabla}\times\mathbf{B}. \tag{23}$$

We evaluate the field components in a cylindrical coordinate system (ρ, ϕ, z) to find for a small antenna with $ka \ll 1$,

$$\begin{aligned}
B_\rho &= 0, && (24)\\
B_\phi &= -Re\left\{\frac{iI_0 e^{-i\omega t}}{c\rho}\left[e^{ikr_1} + e^{ikr_2} - 2e^{ikr_0}\left(1 - \frac{k^2a^2}{2}\right)\right]\right\} && (25)\\
&= \frac{I_0}{c\rho}\left[\sin(kr_1 - \omega t) + \sin(kr_2 - \omega t) - 2\left(1 - \frac{k^2a^2}{2}\right)\sin(kr_0 - \omega t)\right], && (26)\\
B_z &= 0, && (27)\\
E_\rho &= -Re\left\{\frac{iI_0 e^{-i\omega t}}{c\rho}\left[\frac{(z-a)e^{ikr_1}}{r_1} + \frac{(z+a)e^{ikr_2}}{r_2} - 2\frac{ze^{ikr_0}}{r_0}\left(1 - \frac{k^2a^2}{2}\right)\right]\right\} && (28)\\
&= \frac{I_0}{c\rho}\left[(z-a)\frac{\sin(kr_1 - \omega t)}{r_1} + (z+a)\frac{\sin(kr_2 - \omega t)}{r_2} - 2z\left(1 - \frac{k^2a^2}{2}\right)\frac{\sin(kr_0 - \omega t)}{r_0}\right], && (29)\\
E_\phi &= 0, && (30)\\
E_z &= Re\left\{\frac{iI_0 e^{-i\omega t}}{c}\left[\frac{e^{ikr_1}}{r_1} + \frac{e^{ikr_2}}{r_2} - 2\frac{e^{ikr_0}}{r_0}\left(1 - \frac{k^2a^2}{2}\right)\right]\right\} && (31)
\end{aligned}$$

[13]We note that the sequence of calculations in eqs. (22) and (23) could be interpreted as implying that the conduction current I leads to the vector potential and the magnetic field, and then the curl of the magnetic field leads to the "displacement current" $(1/4\pi)\partial E/\partial t$.

$$= -\frac{I_0}{c}\left[\frac{\sin(kr_1-\omega t)}{r_1}+\frac{\sin(kr_2-\omega t)}{r_2}-2\left(1-\frac{k^2a^2}{2}\right)\frac{\sin(kr_0-\omega t)}{r_0}\right]. \tag{32}$$

For completeness, we also display the field components in a spherical coordinate system (r, θ, ϕ), noting that $\rho = r_0 \sin\theta$,

$$B_r = 0, \tag{33}$$

$$B_\theta = 0, \tag{34}$$

$$B_\phi = -Re\left\{\frac{iI_0e^{-i\omega t}}{cr_0}\left[e^{ikr_1}+e^{ikr_2}-2e^{ikr_0}\left(1-\frac{k^2a^2}{2}\right)\right]\sin\theta\right\} \tag{35}$$

$$= \frac{I_0}{cr_0}\left[\sin(kr_1-\omega t)+\sin(kr_2-\omega t)-2\left(1-\frac{k^2a^2}{2}\right)\sin(kr_0-\omega t)\right]\sin\theta, \tag{36}$$

$$E_r = Re\left\{\frac{iaI_0e^{-i\omega t}}{cr_0}\left[\frac{e^{ikr_1}}{r_1}-\frac{e^{ikr_2}}{r_2}\right]\right\} \tag{37}$$

$$= -\frac{I_0a}{cr_0}\left[\frac{\sin(kr_1-\omega t)}{r_1}-\frac{\sin(kr_2-\omega t)}{r_2}\right], \tag{38}$$

$$E_\theta = -Re\left\{\frac{iI_0e^{-i\omega t}}{cr_0^2\sin\theta}\left[\frac{(r_0^2-az)e^{ikr_1}}{r_1}+\frac{(r_0^2+az)e^{ikr_2}}{r_2}-2r_0e^{ikr_0}\left(1-\frac{k^2a^2}{2}\right)\right]\right\} \tag{39}$$

$$= \frac{I_0}{cr_0^2\sin\theta}\left[(r_0^2-az)\frac{\sin(kr_1-\omega t)}{r_1}+(r_0^2+az)\frac{\sin(kr_2-\omega t)}{r_2}\right.$$
$$\left.-2r_0\left(1-\frac{k^2a^2}{2}\right)\sin(kr_0-\omega t)\right]. \tag{40}$$

$$E_\phi = 0, \tag{41}$$

The radiation fields can be found by going to the far zone, where $r = r_0 \approx r_1 \approx r_2$. In spherical coordinates the only nonzero components to the radiation fields are[14]

$$B_\phi = E_\theta = -Re\left\{\frac{iI_0k^2a^2}{c}\frac{e^{i(kr_0-\omega t)}}{r_0}\sin\theta\right\} = \frac{I_0k^2a^2}{c}\frac{\sin(kr_0-\omega t)}{r_0}\sin\theta. \tag{42}$$

The radiation fields depend on the square of the small quantity ka, which permits us to identify the radiation fields in the near zone, where they are only a small part of the total fields. Close to the antenna, where $r_j \ll a$ we have $\cos(kr_j) \approx 1$ and $\sin(kr_j) \approx kr_j$ for $j = 0, 1, 2$. The nonzero field components in cylindrical coordinates close to the antenna are therefore,

$$B_\phi(r_j \ll a) \approx \frac{I_0}{c\rho}\left[k(r_1+r_2-2r_0)\cos(\omega t)-k^2a^2\sin(\omega t)\right], \tag{43}$$

$$E_\rho(r_j \ll a) \approx -\frac{I_0}{c\rho}\left[\frac{z-a}{r_1}+\frac{z+a}{r_2}-\frac{2z}{r_0}\left(1-\frac{k^2a^2}{2}\right)\right]\sin(\omega t), \tag{44}$$

$$E_z(r_j \ll a) \approx \frac{I_0}{c}\left[\frac{1}{r_1}+\frac{1}{r_2}-\frac{2}{r_0}\left(1-\frac{k^2a^2}{2}\right)\right]\sin(\omega t). \tag{45}$$

[14]Verification that eqs. (39) and (40) become eq. (42) in the far zone is a bit subtle. See [12].

Close to the antenna all of the electric field varies as $\sin(\omega t)$, and so is 90° out of phase with the drive current. The largest part of the magnetic field is in phase with the current, but the radiation part of the magnetic field (which includes the factor k^2a^2) is 90° out of phase with the current, and is therefore in phase with the electric field. Furthermore, the radiation parts of the electric and magnetic field have very similar magnitudes close to the antenna, even though the total electric field is much larger than the total magnetic field here.

Thus, a single, short linear dipole antenna has radiation fields in its near zone that are similar in character to the radiation fields in the far zone: $E_{\rm rad} \approx B_{\rm rad}$ in magnitude an phase, and directed at right angles to one another. A single short linear antenna IS a "crossed-field" antenna.

A.2 Appendix B: Fields with Neglect of the Displacement Current

It may be instructive to deduce the electric and magnetic fields for a short linear antenna when the displacement current, $(1/4\pi)\partial\mathbf{E}/\partial t$, is neglected. In this case, Maxwell's equations are

$$\boldsymbol{\nabla}\cdot\mathbf{E} = 4\pi\varrho, \quad \boldsymbol{\nabla}\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \quad \boldsymbol{\nabla}\cdot\mathbf{B} = 0, \quad \text{and} \quad \boldsymbol{\nabla}\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J}. \tag{46}$$

These equations can by satisfied by

$$\mathbf{E} = -\boldsymbol{\nabla}\Phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \qquad \mathbf{B} = \boldsymbol{\nabla}\times\mathbf{A}, \tag{47}$$

where the scalar potential Φ and the vector potential $\mathbf{A}$ are calculated using present quantities,

$$\Phi(\mathbf{x},t) = \int \frac{\varrho(\mathbf{x}',t)}{R}\,d\text{Vol}', \qquad \mathbf{A}(\mathbf{x},t) = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{x}',t)}{R}\,d\text{Vol}'. \tag{48}$$

Of course, this implies that changes in the charge or current distribution cause instantaneous changes in the potentials and fields.

Recalling eqs. (2) and (3) for the charge and current distributions along the antenna,

$$\varrho(z<|a|,t) = \pm\frac{iI_0e^{-i\omega t}}{a\omega} \qquad I(z<|a|,t) = I_0e^{-i\omega t}\left(1-\frac{|z|}{a}\right), \tag{49}$$

we see that the scalar potential Φ is 90° out of phase with the vector potential $\mathbf{A}$. The time derivative $\partial\mathbf{A}/\partial t$ is 90° out of phase with $\mathbf{A}$, and hence is in phase with Φ. Then, eq. (47) indicates that the electric and magnetic fields are 90° out of phase throughout all space, IF the "displacement current" is neglected.

In detail, we find in cylindrical coordinates that

$$\begin{aligned}\Phi(\rho,\phi,z,t) &= Re\frac{iI_0e^{-i\omega t}}{a\omega}\left(\int_0^a \frac{dz'}{\sqrt{\rho^2+(z-z')^2}} - \int_{-a}^0 \frac{dz'}{\sqrt{\rho^2+(z-z')^2}}\right) \\ &= \frac{I_0}{a\omega}\sin(\omega t)\left\{\ln\left[\sqrt{\rho^2+(z-a)^2}-(z-a)\right] + \ln\left[\sqrt{\rho^2+(z+a)^2}-(z+a)\right]\right. \\ &\qquad \left. -2\ln\left[\sqrt{\rho^2+z^2}-z\right]\right\},\end{aligned} \tag{50}$$

and

$$\begin{aligned}
A_z(\rho, \phi, z, t) &= Re \frac{I_0}{c} e^{-i\omega t} \left(\int_0^a \frac{(1 - z'/a)\, dz'}{\sqrt{\rho^2 + (z - z')^2}} + \int_{-a}^0 \frac{(1 + z'/a)\, dz'}{\sqrt{\rho^2 + (z - z')^2}} \right) \\
&= \frac{I_0}{c} \cos(\omega t) \left\{ \left(1 - \frac{z}{a}\right) \ln \left[\sqrt{\rho^2 + (z-a)^2} - (z - a) \right] \right. \\
& \qquad - \left(1 + \frac{z}{a}\right) \ln \left[\sqrt{\rho^2 + (z+a)^2} - (z + a) \right] + \frac{2z}{a} \ln \left[\sqrt{\rho^2 + z^2} - z \right] \\
& \qquad \left. - \frac{1}{a} \left[\sqrt{\rho^2 + (z-a)^2} + \sqrt{\rho^2 + (z+a)^2} - 2\sqrt{\rho^2 + z^2} \right] \right\}. \qquad (51)
\end{aligned}$$

Far from the antenna the potentials (50) and (51) simplify to the forms

$$\Phi \approx \frac{I_0 a}{\omega} \frac{z}{r_0^3} \sin(\omega t), \qquad \text{and} \qquad A_z \approx \frac{I_0 a}{r_0 c} \cos(\omega t). \qquad (52)$$

The scalar potential is that of a dipole consisting of charges $\pm i I_0/\omega$ separated by distance a, and the vector potential is that due to a length a of current I_0, with both potentials oscillating at frequency ω.

The magnetic field again has only a ϕ component, but now it varies only as $\cos(\omega t)$,

$$\begin{aligned}
B_\phi &= -\frac{\partial A_z}{\partial \rho} = \frac{\rho I_0}{ac} \cos(\omega t) \left[\frac{1}{r_1 - (z - a)} + \frac{1}{r_2 - (z + a)} - \frac{2}{r_0 - z} \right] \\
&\approx \frac{I_0 a \sin\theta}{r_0^2 c} \cos(\omega t), \qquad (53)
\end{aligned}$$

where the approximation holds for $r_0 \gg a$. The ρ component of the electric field is

$$\begin{aligned}
E_\rho &= -\frac{\partial \Phi}{\partial \rho} = -\frac{\rho I_0}{a\omega} \sin(\omega t) \left[\frac{1}{r_1(r_1 - (z - a))} + \frac{1}{r_2(r_2 - (z + a))} - \frac{2}{r_0(r_0 - z)} \right] \\
&\approx \frac{3 I_0 a \cos\theta \sin\theta}{\omega r_0^3} \sin(\omega t), \qquad (54)
\end{aligned}$$

and z component of the electric field is

$$\begin{aligned}
E_z &= -\frac{\partial \Phi}{\partial z} - \frac{1}{c}\frac{\partial A_z}{\partial t} = \frac{I_0}{a\omega} \sin(\omega t) \left\{ \frac{1}{r_1} + \frac{1}{r_2} - \frac{2}{r_0} \right. \\
& \qquad \left. - k^2 [r_1 + r_2 - 2r_0 + (z - a) \ln(r_1 - z + a) + (z + a) \ln(r_2 - z - a) - 2z \ln(r_0 - z)] \right\} \\
&\approx \frac{I_0 a}{\omega} \left(\frac{3\cos^2\theta - 1}{r_0^3} - \frac{k^2}{r_0} \right) \sin(\omega t). \qquad (55)
\end{aligned}$$

The electric field varies only as $\sin(\omega t)$.

The components of the electric field in spherical coordinates for $r_0 \gg a$ are

$$E_r \approx \frac{I_0 a}{\omega} \left(\frac{2}{r_0^3} - \frac{k^2}{r_0} \right) \cos\theta \sin(\omega t), \qquad E_\theta \approx \frac{I_0 a}{\omega} \left(\frac{1}{r_0^3} + \frac{k^2}{r_0} \right) \sin\theta \sin(\omega t). \qquad (56)$$

The fields in both the near and the far zones are very different in the case that the "displacement current" is neglected compared to the case when it is properly included. That is, the effect of the "displacement current" is not small in situations where radiation is present, and one gains little insight from an approximation that disregards the "displacement current".[15]

A.3 Appendix C: The Electric and Magnetic Fields are Not Just Retarded Static Fields

It is well known that the electric and magnetic fields described by Maxwell's equations can be deduced from the retarded potentials [14],

$$\Phi(\mathbf{x},t) = \int \frac{\varrho(\mathbf{x}',t')}{R}\, d\mathrm{Vol}', \qquad \mathbf{A}(\mathbf{x},t) = \frac{1}{c}\int \frac{\mathbf{J}(\mathbf{x}',t')}{R}\, d\mathrm{Vol}'. \tag{57}$$

which have the form of the static potentials (48), but with the charge and current distributions evaluated at the retarded time $t' = t - R/c$, where $R = |\mathbf{x} - \mathbf{x}'|$, rather than at the present time. However, it does NOT follow that the electric and magnetic fields have the form of the static fields with the charge and current distributions evaluated at the retarded time. Instead, the fields can be calculated from the charge and current distributions according to[16]

$$\mathbf{E} = \int \frac{[\varrho]\,\hat{\mathbf{R}}}{R^2}\, d\mathrm{Vol}' + \frac{1}{c}\int \frac{([\mathbf{J}]\cdot\hat{\mathbf{R}})\hat{\mathbf{R}} + ([\mathbf{J}]\times\hat{\mathbf{R}})\times\hat{\mathbf{R}}}{R^2}\, d\mathrm{Vol}' + \frac{1}{c^2}\int \frac{([\dot{\mathbf{J}}]\times\hat{\mathbf{R}})\times\hat{\mathbf{R}}}{R}\, d\mathrm{Vol}', \tag{58}$$

and

$$\mathbf{B} = \frac{1}{c}\int \frac{[\mathbf{J}]\times\hat{\mathbf{R}}}{R^2}\, d\mathrm{Vol}' + \frac{1}{c^2}\int \frac{[\dot{\mathbf{J}}]\times\hat{\mathbf{R}}}{R}\, d\mathrm{Vol}', \tag{59}$$

where $\hat{\mathbf{R}} = \mathbf{R}/R = (\mathbf{x} - \mathbf{x}')/\,|\mathbf{x} - \mathbf{x}'|$, and quantities inside brackets, [...], are evaluated at the retarded time $t' = t - R/c$.

If the charge and current distributions are oscillatory with a single frequency ω, we can write

$$\varrho(\mathbf{x},t) = \varrho_0(\mathbf{x})e^{-i\omega t}, \qquad \text{and} \qquad \mathbf{J}(\mathbf{x},t) = \mathbf{J}_0(\mathbf{x})e^{-i\omega t}. \tag{60}$$

The oscillatory factor $e^{-i\omega t}$ when evaluated at the retarded time $t' = t - R/c$ becomes the waveform $e^{-i\omega(t'-R/c)} = e^{i(kR-\omega t)}$, where $k = \omega/c = 2\pi/\lambda$. In this case, the electric and magnetic fields can be written as

$$\begin{aligned}\mathbf{E} \;&=\; \int \frac{\varrho_0\hat{\mathbf{R}}}{R^2} e^{i(kR-\omega t)}\, d\mathrm{Vol}' + \frac{1}{c}\int \frac{(\mathbf{J}_0\cdot\hat{\mathbf{R}})\hat{\mathbf{R}} + (\mathbf{J}_0\times\hat{\mathbf{R}})\times\hat{\mathbf{R}}}{R^2} e^{i(kR-\omega t)}\, d\mathrm{Vol}' \\ &\quad - \frac{ik}{c}\int \frac{(\mathbf{J}_0\times\hat{\mathbf{R}})\times\hat{\mathbf{R}}}{R} e^{i(kR-\omega t)}\, d\mathrm{Vol}', \end{aligned} \tag{61}$$

[15]It might be argued that since the "displacement current" leads to radiation, which is a small effect in the near zone, the fields in the near zone should be almost the same whether or not the "displacement current" is taken into account. The present example shows that this is not the case.

[16]Equations (58) and (59) first appeared in [7].

and

$$\mathbf{B} = \frac{1}{c} \int \frac{\mathbf{J}_0 \times \hat{\mathbf{R}}}{R^2} e^{i(kR-\omega t)}\, d\mathrm{Vol}' - \frac{ik}{c} \int \frac{\mathbf{J}_0 \times \hat{\mathbf{R}}}{R} e^{i(kR-\omega t)}\, d\mathrm{Vol}', \tag{62}$$

The first term of eqs. (58) and (61) could be called the retarded Coulomb field, and the first term of eqs. (59) and (62) could be called the retarded Biot-Savart field. Both of these terms vary as the inverse square of the distance between the source and observer, and so they are important in the near zone and negligible in the far zone.

It is perhaps surprising that the electric field has a second term that varies inversely with the square of the distance, and which is due to the current distribution rather than the charge distribution.[17] This term is an indirect effect of Maxwell's "displacement current", and in examples such as the present it makes a significant contribution to the difference between the actual near-zone fields and those approximated by neglect of the "displacement current".

The last terms of eqs. (58)-(59) and (61)-(62) vary inversely with the distance between the source and observer. These terms are the radiation fields, which are the most significant additions to the fields when the "displacement current" is included in Maxwell's equations.

The form of these terms shows that each current element whose time derivative is nonzero creates electric and magnetic radiation fields that are 90° out of phase with respect to the current, and which are equal in magnitude and at right angles to one another. If the current elements are in phase, and their spatial extent is small compared to a wavelength, the radiation fields from different current elements are in phase with respect to one another, and there is constructive interference between them. Only if the radiator is large compared to a wavelength can the total strength of the radiated fields be increased by introducing phase differences between current elements.

A.4 Appendix D: Designer Near Fields for "Small" Antennas

For "small" antennas, whose size is much less than a wavelength, the far-field radiation pattern can only be that of a Hertzian dipole [16, 17]. More complex far-field radiation patterns arise only if the size of the antenna is comparable to (or larger than) a wavelength, such that effects of retardation between different components of the antenna become important.

Here, we restrict our attention to "small" antennas, and consider what amount of variation of near fields is possible, consistent with the same far field radiation pattern.

We shall distinguish two subregions of the near field. If the antenna has characteristic length a, and radiates waves of length $\lambda \gg a$, the radiation fields become larger than the quasistatic fields only for distances $\gtrsim \lambda$ from the antenna. The "near zone" is the region in which the radiation fields are not yet prominent, and so is the region within distance λ of the antenna.

Close to the conductors of the antenna, the details of the fields are very dependent on the geometry of the conductors. However, at distances $\gtrsim 2a$ from the antenna the fields take on the form of an ideal Hertzian dipole radiator.

[17]The second term of the electric field vanishes for steady currents. See sec. 3 of [15]. While this term is expressed as a function only of the conduction currents, it would be absent if the "displacement current" were not present in Maxwell's equations.

In designing the near fields of an antenna, we therefore should consider separately what forms are possible in the region $\lesssim 2a$ from the antenna, and the region from $\approx 2a$ to $\approx \lambda$ from the antenna.

The options in the latter region are much more restricted than in the former, so we consider the latter case first.

We recall that there are two forms of Hertzian dipole radiators, electric dipoles and magnetic dipoles.[18]

Electric dipole radiators that broadcast at angular frequency ω are characterized by their electric dipole moment $\mathbf{p}e^{-i\omega t}$ where vector $\mathbf{p}$ is constant in time but can have complex components. Similarly, magnetic dipole radiators are characterized by their magnetic dipole moment $\mathbf{m}e^{-i\omega t}$, where the constant vector $\mathbf{m}$ can have complex components.

The electromagnetic fields of these electric and magnetic dipole radiators are, for distances $\gtrsim 2a$ from the radiator (whose size is a) are (in Gaussian units) [17]

$$\begin{aligned}
\mathbf{E} &= k^2[(\hat{\mathbf{r}} \times \mathbf{p}) \times \hat{\mathbf{r}} - \hat{\mathbf{r}} \times \mathbf{m}]\frac{e^{i(kr-\omega t)}}{r} - ik\{[3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}] - \hat{\mathbf{r}} \times \mathbf{m}\}\frac{e^{i(kr-\omega t)}}{r^2} \\
&\quad +[3\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]\frac{e^{i(kr-\omega t)}}{r^3}, \qquad (63) \\
\mathbf{B} &= k^2[(\hat{\mathbf{r}} \times \mathbf{m}) \times \hat{\mathbf{r}} + \hat{\mathbf{r}} \times \mathbf{p}]\frac{e^{i(kr-\omega t)}}{r} - ik\{[3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}] - \hat{\mathbf{r}} \times \mathbf{p}\}\frac{e^{i(kr-\omega t)}}{r^2} \\
&\quad +[3\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]\frac{e^{i(kr-\omega t)}}{r^3}, \qquad (64)
\end{aligned}$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector from the center of the dipole to the observer,

The only flexibility we have in the design of these fields are our choices as to the magnitudes, directions and phases of the magnetic moments $\mathbf{p}$ and $\mathbf{m}$.

In the near field, where $r < \lambda$, the terms in eqs. (63) and (64) that vary as $1/r^3$ are the largest. That is,

$$\mathbf{E}_{\text{near}}(2a \lesssim r \lesssim \lambda) \approx [3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}]\frac{e^{i(kr-\omega t)}}{r^3}, \qquad (65)$$

$$\mathbf{B}_{\text{near}}(2a \lesssim r \lesssim \lambda) \approx [3(\mathbf{m} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}]\frac{e^{i(kr-\omega t)}}{r^3}. \qquad (66)$$

These fields have the shape of static dipole fields multiplied by the traveling wave $e^{i(kr-\omega t)}$, and thereby have components both parallel to and transverse to the radial direction, in contrast to the radiation fields that are purely transverse. Note that the electric field in the near zone is, in the first approximation, due only to the electric dipole antenna, while the magnetic field in the near zone is due only to the magnetic dipole antenna. Hence, no combination of small electric and magnetic dipole antennas can eliminate the nonradiating fields in the near zone, as may be a goal of enthusiasts for "crossed-field" antennas.

[18] Actually, there is a third possible form of small antennas, the so-called helical toroidal dipole antenna [18], aspects of which may be (unknowingly) incorporated into the design of "cross-field" antennas such as that of [3]. However, unless helical toroidal antennas involve counter windings, they are in effect single-turn loop antennas, as considered here.

If we desire the electric and magnetic fields (65)-(66) to be equal in magnitude in the near zone to a first approximation, then we need $|\mathbf{m}| = |\mathbf{p}|$.[19]

If in addition, we desire the electric and magnetic fields to be 90° out of phase in the near zone, we need $\mathbf{m} = i\,|\mathbf{p}|\ \hat{\mathbf{m}}$, where the directions $\hat{\mathbf{m}}$ and $\hat{\mathbf{p}}$ are arbitrary.

It is not possible to satisfy the preceding constraints and have the electric and magnetic fields everywhere at right angles to one another in the near field. If these fields were at right angles, their scalar product,

$$\mathbf{E}_{\rm near} \cdot \mathbf{B}_{\rm near} \propto 3(\mathbf{m} \cdot \hat{\mathbf{r}})(\mathbf{p} \cdot \hat{\mathbf{r}}) + \mathbf{m} \cdot \mathbf{p}, \tag{67}$$

should vanish. Consider a coordinate system with $\mathbf{p}$ *along the* z*-axis. Then, vector* $\mathbf{m}$ *points along angles* (θ_m, ϕ_m) *in spherical coordinates, and has rectangular coordinates*

$$\mathbf{m} = m(\sin\theta_m \cos\phi_m, \sin\theta_m \sin\phi_m, \cos\theta_m). \tag{68}$$

The radial unit vector has components

$$\hat{\mathbf{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{69}$$

Hence,

$$\mathbf{E}_{\rm near} \cdot \mathbf{B}_{\rm near} \propto 3[\sin\theta\sin\theta_m\cos(\phi - \phi_m) + \cos\theta\cos\theta_m]\cos\theta + \cos\theta_m, \tag{70}$$

which cannot vanish for all θ *and* ϕ *for any choice of* θ_m *and* ϕ_m.

Similarly, the transverse parts of the near electric and magnetic fields cannot be at right angles to one another everywhere.

We close by considering radiation from a combination of a small electric and small magnetic antenna with common centers, taken to be the origin. The radiation fields have the same form for any $r \gtrsim 2a$, which region includes most of the near zone and all of the far zone,

$$\mathbf{E}_{\rm rad}(r \gtrsim 2a) = k^2[(\hat{\mathbf{r}} \times \mathbf{p}) \times \hat{\mathbf{r}} - \hat{\mathbf{r}} \times \mathbf{m}]\frac{e^{i(kr-\omega t)}}{r} = k^2[\mathbf{p} - (\hat{\mathbf{r}} \cdot \mathbf{p})\hat{\mathbf{r}} - \hat{\mathbf{r}} \times \mathbf{m}]\frac{e^{i(kr-\omega t)}}{r} \tag{71}$$

$$\mathbf{B}_{\rm rad}(r \gtrsim 2a) = k^2[(\hat{\mathbf{r}} \times \mathbf{m}) \times \hat{\mathbf{r}} + \hat{\mathbf{r}} \times \mathbf{p}]\frac{e^{i(kr-\omega t)}}{r} = k^2[\mathbf{m} - (\hat{\mathbf{r}} \cdot \mathbf{m})\hat{\mathbf{r}} - \hat{\mathbf{r}} \times \mathbf{p}]\frac{e^{i(kr-\omega t)}}{r} \tag{72}$$

The time-average radiated power has the angular distribution[20]

$$\left\langle \frac{dP(\hat{\mathbf{r}})}{d\Omega} \right\rangle = \frac{cr^2}{8\pi}\hat{\mathbf{r}} \cdot Re(\mathbf{E} \times \mathbf{B}^\star) = \frac{ck^4}{8\pi}\left(|p|^2 \sin^2\theta_p + |m|^2 \sin^2\theta_m\right), \tag{73}$$

where θ_p is the angle between $\hat{\mathbf{r}}$ and $\mathbf{p}$, and θ_m is the angle between $\hat{\mathbf{r}}$ and $\mathbf{m}$. A possibly surprising result is that there is no interference between the radiation from the electric dipole

[19] To have equality of electric and magnetic fields in the near zone we must have both electric and magnetic antennas. The use of two electric antennas with moments $\mathbf{p}_1$ and $\mathbf{p}_2$, as advocated in one design of a "crossed-field" antenna [2], merely leads to an electric antenna of total moment $\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$, for which the near electric field is always larger than the near magnetic field.

[20] $\hat{\mathbf{r}} \cdot [\mathbf{p} - (\hat{\mathbf{r}} \cdot \mathbf{p})\hat{\mathbf{r}}] \times [\mathbf{m}^\star - (\hat{\mathbf{r}} \cdot \mathbf{m}^\star)\hat{\mathbf{r}}] = \hat{\mathbf{r}} \cdot \mathbf{p} \times \mathbf{m}^\star$, while $\hat{\mathbf{r}} \cdot (\hat{\mathbf{r}} \times \mathbf{m}) \times (\hat{\mathbf{r}} \times \mathbf{p}^\star) = -(\hat{\mathbf{r}} \times \mathbf{m}) \cdot \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{p}^\star) = -(\hat{\mathbf{r}} \times \mathbf{m}) \cdot [(\hat{\mathbf{r}} \cdot \mathbf{p}^\star)\hat{\mathbf{r}} - \mathbf{p}^\star] = -\hat{\mathbf{r}} \cdot \mathbf{p}^\star \times \mathbf{m}$, so the sum of these two terms has no real part.

p and the magnetic dipole **m**, no matter what are their directions and relative phases. The total time-average radiated power follows from integration of eq. (73),

$$\langle P \rangle = \frac{ck^4}{3}\left(|p|^2 + |m|^2\right) = P_E + P_M, \tag{74}$$

where P_E and P_M are the time-average powers radiated by the small electric and magnetic antennas if operated separately. Thus, there is no advantage (in terms of radiated power) to a combination of a small electric dipole and a small magnetic dipole antenna compared to either of these two separately.[21]

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