Induction from the Magnetic Vector Potential Field

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1. Introduction

Movement of charge q through a magnetic vector potential **A** field is known to create a force **F** on the charge when the vector **A** possesses curl. This is usually expressed by the $\frac{\mathbf{F}}{q} = \mathbf{E} = (\mathbf{v} \times \mathbf{B})$ motional induction, although the full expression is $\mathbf{E} = (\mathbf{v} \times (\nabla \times \mathbf{A}))$ where **v** is the velocity. Of interest is the fact that there are no longitudinal (along the velocity direction) components of force. There is much debate about whether motional induction can occur in a non-curl **A** field where $\mathbf{B} = 0$, that is the field that exists external to flux carried within a core. Certainly when that flux (hence also the external **A** field) changes with time there is classical transformer induction via that external **A** field, usually expressed by $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$. The fact that this electric field can only deliver voltage to a closed circuit that encircles the flux-carrying core has led to the belief that any closed circuit that does not encircle the core cannot obtain induction. To challenge that belief it is necessary to fully understand the argument that says if the **A** field changes with distance (i.e. is not uniform) any moving charge will "see" a time-changing **A** field, with the rate-of-change proportional to its velocity. If that is generally the case then it should be possible to express classical $\mathbf{E} = (\mathbf{v} \times \mathbf{B})$ motional induction in terms of that time-changing **A** field "seen" by the moving charge.

In this paper we take the well-known situation of a very long magnetized cylindrical core where the internal **B** field is uniform and externally **B** is zero. Both the internal and external **A** field vectors form concentric circles that can be expressed with simple math. Consideration is given to charge moving in a plane normal to the cylinder axis. Starting with the internal fields, the changing **A** field seen by the moving charge is shown to produce the classical $\mathbf{E} = (\mathbf{v} \times \mathbf{B})$ induction only if account is taken of the two separate effects,

(a) change of the **A** vector *amplitude* with charge movement and
(b) change of the **A** vector *orientation* with charge movement.

It may be noted that (b) the second of these has generally been ignored in the past, but the fact that it is necessary to correctly reproduce one of the most tested laws in physics shows the importance of including this.

That a time-changing **A** vector *orientation* should have an effect on a charge q should be obvious when it is realized that the vector $q\mathbf{A}$ is a form of momentum, and q will endure a force whenever that momentum changes. It is instructive to look at classical mass momentum to see how change of *orientation* of an otherwise fixed amplitude momentum-vector creates a force on the mass. This occurs when a mass m moves at a constant speed v in a circular movement, giving rise to centrifugal force. Although the momentum magnitude mv is constant the momentum vector $\mathbf{p} = m\mathbf{v}$ rotates at angular velocity ω giving rise to a centrifugal force $\omega\mathbf{p}$ at right angles to **p**. Also any movement of the mass m within the centrifuge yields a Coriolis force at right angles to that movement. Within the rotating reference frame the mass m inherits its momentum $m\mathbf{v}$, and within the circular **A** field the charge q inherits its momentum $q\mathbf{A}$. For the internal **A** field it is necessary to include both the centrifugal-like and Coriolis-like forces on the charge q. Having established the correct procedures for deriving all the force components from the internal **A** field, those procedures are then applied to the external **A** field where **B** is zero.

2. A field internal to the core.

Figure 1 shows one quadrant of a cross section of the core that carries the **B** field, this field points out of the screen. The **A** field forms concentric circles around the centre. Because the closed line integral around **A** at any fixed radius R equals the total flux passing through that circle we can say $2\pi RA = \pi R^2 B$ from which

$$|A| = \frac{BR}{2} \tag{1}$$

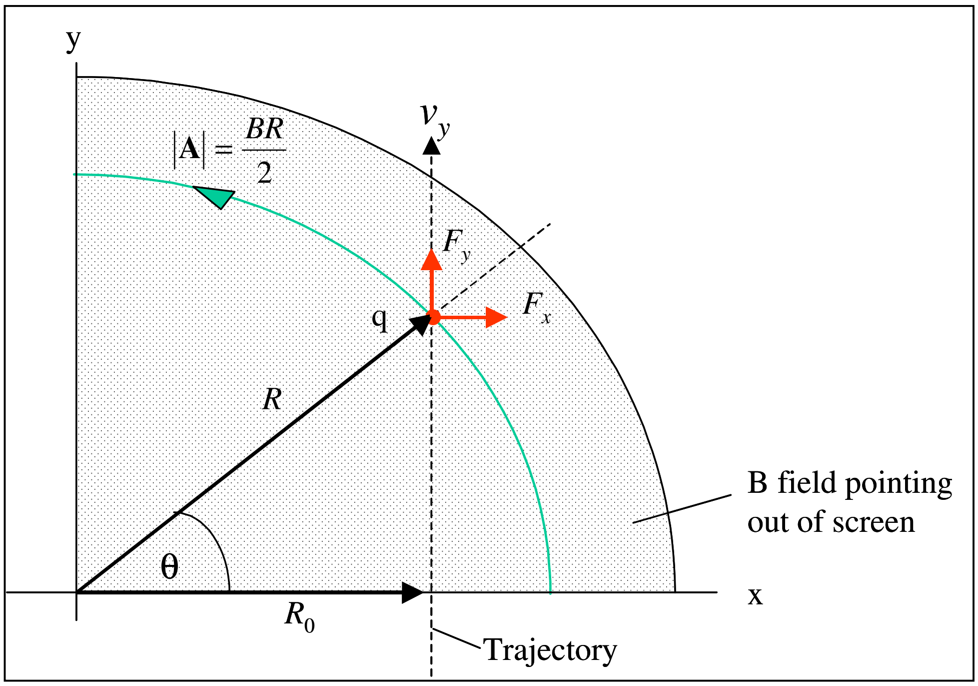

Figure 1. Charge q moving across a B field

We select a charge q moving in the y direction at velocity v_y passing by the core centre with a miss distance R_0. At any point y along the trajectory the A field at q has a radius R given by $R = \sqrt{R_0^{\ 2} + y^2}$. At that point the y component of the A field is $A_y = A\cos\theta = \frac{AR_0}{R} = \frac{BR_0}{2}$ which is constant. Thus $\frac{\partial A_y}{\partial y} = 0$ hence

$$E_y = -v_y \frac{\partial A_y}{\partial y} = 0, \tag{2}$$

there is no longitudinal induction.

At that point the x component of the A field is $A_x = -A\sin\theta = -\frac{Ay}{R} = -\frac{By}{2}$. Thus $\frac{\partial A_x}{\partial y} = -\frac{B}{2}$ and

$$E_x = -v_y \frac{\partial A_x}{\partial y} = v_y \frac{B}{2}. \tag{3}$$

This is only half the known value as given by the $\mathbf{E} = (\mathbf{v} \times \mathbf{B})$ motional induction.

The time-changing **A** from terms like $v_y \frac{\partial A_x}{\partial y}$ in (2) and (3) take note of amplitude changes in the vector **A**, but changes in vector orientation can also yield an electric field. Consider a vector **A** of constant amplitude but it is rotating about the z axis at an angular rate ω. If we plot x and y components of **A** against time we get sinusoidal values with a 90 degree phase difference. Differentiating those plots to get $\frac{\partial A}{\partial t}$ yields a 90 degree shift of both plots, then the resulting waveforms represent a rotating vector at 90 degrees to the original **A**, figure 2.

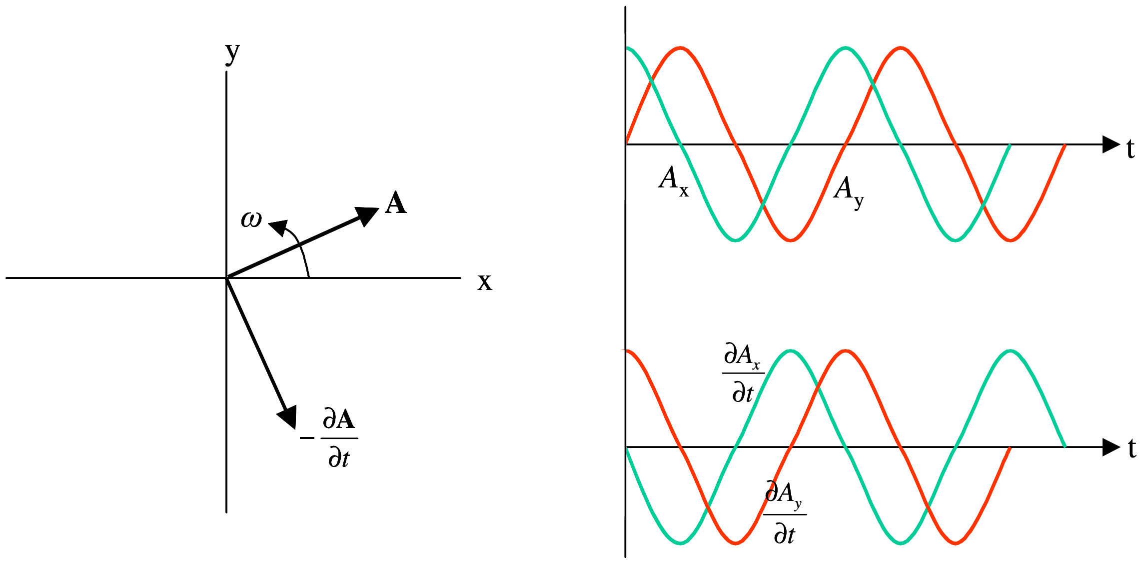

Figure 2. Rotating A vector

The resulting **E** vector from $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t}$ is seen to lag the revolving **A** vector and its amplitude is given by

$$|\mathbf{E}| = \omega |\mathbf{A}|. \qquad (4)$$

Now consider a charge q that follows a circular path at radius R and (tangential) velocity v_t so that it sees a constant value **A** field of magnitude $BR/2$. Not only is the magnitude constant but also the alignment of the velocity **v** vector with the **A** vector, hence any longitudinal $\frac{\partial A}{\partial t}$ is also apparently absent. But in fact the direction of **A** is constantly changing as depicted in Figure 2. Its angular frequency is given by $\omega = \frac{v_t}{R}$ which when put into (4) and using (1) gives

$$E_r = \omega \frac{A}{R} = v_t \frac{B}{2} \qquad (5)$$

This is at right angles to **A** as in Figure 2 hence results in a radial force on q. This may be likened to centrifugal force that is also radial. The circular **A** field confers momentum onto the charge q in a similar way that a rotating reference frame confers momentum onto a mass m. Perhaps this is not surprising as the **A** field emanates from rotating charges that are its source. It is instructive to look at Mach's principle that implies mass inertia comes from some form of interaction with distant matter, and you would obtain the same centrifugal force if the mass m were stationary and the Universe revolved around it. Actually that is not quite correct, the mass m translates through space in a small circular orbit, and the same centrifugal force would be obtained if the mass m were stationary and the Universe translated around a small orbit, i.e. each mass followed a motion like that of the face-plate of an orbital sander. In a way this is exactly what we have with the charges responsible for the magnetic field, they follow tiny orbits mimicking the movement of the grains of sand on the orbital sander, and that movement is responsible for the circular field influence on our test charge q. With that in mind we must also consider the equivalent to Coriolis forces on mass within a rotating frame, where a radial movement gives rise to a tangential force. In our system this yields

$$E_t = v_r \frac{B}{2} \tag{6}$$

We now show that including (5) and (6) for the charge movement shown in Figure 1 adds the missing force to (3) and we have full compatibility with classical motional induction $\frac{\mathbf{F}}{q} = \mathbf{E} = (\mathbf{v} \times \mathbf{B})$.

Looking again at Figure 1 we see that tangential velocity $v_t = v_y \cos\theta = \frac{v_y R_0}{R}$. Then using (5) we get a radial **E** field $E_r = \frac{v_y B \cos\theta}{2}$ and that gives rise to an E_x component $\delta E_x = E_r \cos\theta = \frac{v_y B \cos^2\theta}{2}$. The radial velocity $v_r = v_y \sin\theta$ put into (6) yields a tangential **E** field $E_t = -\frac{v_y B \sin\theta}{2}$ giving rise to an E_x component $\delta E_x = -E_t \sin\theta = \frac{v_y B \sin^2\theta}{2}$. Adding both E_x components yields $E_x = \frac{v_y B}{2}(\sin^2\theta + \cos^2\theta) = \frac{v_y B}{2}$. This when added to (3) doubles the value to agree exactly with that from $\mathbf{E} = (\mathbf{v} \times \mathbf{B})$.

Dong the same procedure for E_y we obtain $\delta E_y = E_r \sin\theta = \frac{v_y B}{2} \sin\theta\cos\theta$ and $\delta E_y = E_t \cos\theta = -\frac{v_y B}{2} \sin\theta\cos\theta$ that when added equate to zero. Thus the Coreolis-like forces do not contribute to longitudinal induction.

3. A field external to core

Figure 3 shows the **A** field outside the core carrying flux Φ. The **A** field forms circles concentric to the axis, and from the known fact that the closed line integral of **A** equals the flux Φ we can state $2\pi RA = \Phi$ where R is the radius at the chosen point, hence

$$|A| = \frac{\Phi}{2\pi R} \tag{7}$$

As before we select a charge q moving in the y direction at velocity v_y passing by the core centre with a miss distance R_0. At any point y along the trajectory the A field at q has a radius R given by $R = \sqrt{R_0^2 + y^2}$. At that point the y component of the **A** field is $A_y = A\cos\theta = \frac{AR_0}{R} = \frac{\Phi R_0}{2\pi R^2} = \frac{\Phi R_0}{2\pi(R_0^2 + y^2)}$. Taking the differential of A_y with respect to y yields $\frac{\partial A_y}{\partial y} = -\frac{\Phi R_0 y}{\pi(R_0^2 + y^2)^2}$. Hence E_y is given by

$$E_y = -v_y \frac{\partial A_y}{\partial y} = \frac{v_y \Phi R_0 y}{\pi (R_0{}^2 + y^2)^2}. \tag{8}$$

This is a longitudinal (along the velocity direction) induction, but we have yet to establish whether Coreolis-like forces will add to this.

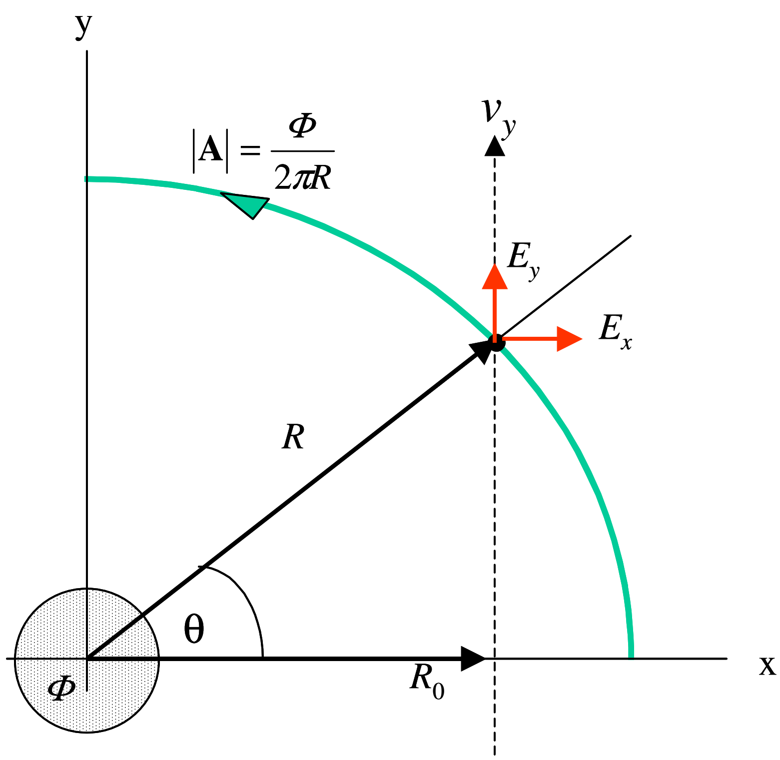

Figure 3. Charge *q* moving external to the B field.

Turning to the transverse induction, the *x* component of the **A** field is $A_x = -A\sin\theta = -\frac{Ay}{R} = -\frac{\Phi y}{2\pi R^2} = -\frac{\Phi y}{2\pi (R_0{}^2 + y^2)}$. Taking the differential of A_x with respect to *y* yields $\frac{\partial A_x}{\partial y} = \frac{\Phi}{2\pi}\left(\frac{2y^2}{(R_0{}^2 + y^2)^2} - \frac{1}{(R_0{}^2 + y^2)}\right)$. Hence E_x is given by

$$E_x = -v_y \frac{\partial A_x}{\partial y} = \frac{v_y \Phi}{2\pi}\left(\frac{1}{(R_0{}^2 + y^2)} - \frac{2y^2}{(R_0{}^2 + y^2)^2}\right) \tag{9}$$

but again we have yet to establish whether Coreolis-like forces will add to this.

Looking at Figure 3 we see that tangential velocity $v_t = v_y \cos\theta = \frac{v_y R_0}{R}$. Then using (5) we get a radial **E** field $E_r = \frac{v_t A}{R} = \frac{v_t \Phi}{2\pi R^2} = \frac{v_y \Phi R_0}{2\pi R^3}$ and that gives rise to an E_x component $\delta E_x = E_r \cos\theta = \frac{v_y \Phi R_0{}^2}{2\pi R^4}$. The radial velocity $v_r = v_y \sin\theta$ yields a tangential **E** field $E_t = -\frac{v_r A}{R} = -\frac{v_r \Phi}{2\pi R^2} = -\frac{v_y \Phi y}{2\pi R^3}$ giving rise to an E_x component $\delta E_x = -E_t \sin\theta = \frac{v_y \Phi y^2}{2\pi R^4}$. Adding both δE_x components yields $E_x = \frac{v_y \Phi}{2\pi R^2}$ as the Coreolis-like contribution to be added to (9). Thus the total transverse field is

$$E_x = \frac{v_y \Phi}{\pi}\left(\frac{1}{(R_0{}^2 + y^2)} - \frac{y^2}{(R_0{}^2 + y^2)^2}\right) \tag{10}$$

Dong the same procedure for δE_y we obtain $\delta E_y = E_r \sin\theta = \frac{v_y \Phi R_0 y}{2\pi R^4}$ and $\delta E_y = E_t \cos\theta = -\frac{v_y \Phi R_0}{2\pi R^4}$ that when added equate to zero. Thus again the Coreolis-like forces do not contribute to longitudinal induction.

Longitudinal and transverse inductions (8) and (10) can be expressed in a different form as

$$E_y = \frac{v_y \Phi}{R_0{}^2}(\cos^3\theta \sin\theta) \tag{11}$$

and

$$E_x = \frac{v_y \Phi}{R_0{}^2}(\cos^4\theta) \tag{12}$$

where $\theta = \tan^{-1}\left(\frac{y}{R_0}\right)$.

4. Conclusion

By taking account of both the change of vector *amplitude* and the change of vector *orientation* with distance it is possible to obtain classical $\mathbf{E} = (\mathbf{v} \times \mathbf{B})$ motional induction directly from the time-rate-of-change of the magnetic vector potential **A**. When this procedure is applied to a non-curl **A** field where **B** is zero it predicts both a longitudinal (8) or (11) and a transverse (10) or (12) induced force on moving charge.