Electromagnetic potentials basis for energy density and power flux

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It is well understood that various alternatives are available within EM theory for the definitions of energy density, momentum transfer, EM stress-energy tensor, and so forth. Although the various options are all compatible with the basic equations of electrodynamics (e.g., Maxwell's equations, Lorentz force law, gauge invariance), nonetheless certain alternative formulations lend themselves to being seen as preferable to others with regard to the transparency of their application to physical problems of interest.

One can argue that the standard formulation encountered in textbooks (and in mainstream use) for energy density and power flux,

$$u(\mathbf{r},t) = \frac{1}{2}\varepsilon_0\left[|\mathbf{E}|^2 + c^2|\mathbf{B}|^2\right] \quad (1.a), \quad \mathbf{S}(\mathbf{r},t) = \varepsilon_0 c^2 (\mathbf{E}\times\mathbf{B}) \quad (1.b)$$

even though resulting in paradoxes, owes its staying power more to historical development than to transparency of application. One oft-noted paradox in the literature, for example, is the (mathematical) apparency of unobservable momentum transfer at a point in static superposed electric and magnetic fields,[1] the consequences of which have on occasion led to time-consuming debate as to the feasibility of certain forms of electromagnetic propulsion. Though such EM foundation issues have been addressed *ad hoc* in the literature, including in this journal,[1,2] a well-organized and systematic treatment that has much to recommend it is as given, for example, in a book by Ribaric and Sustersic (hereafter R&S).[3] Outlined herein are a few of the main points in the recommended approach that provide a welcome transparency.

Although the vector and scalar potentials $(\mathbf{A},\varphi)$ are considered simply as an option in classical theory, in quantum theory they are understood to be more fundamental than the derivative electric and magnetic fields $(\mathbf{E},\mathbf{B})$ which are the "coin of the realm" in ordinary classical theory.[4] In classical electrodynamics the choice of which variable pair to use is arbitrary, and the overall resulting predictions in terms of observables are indistinguishable. Nonetheless, cogent arguments can be made that the $(\mathbf{A},\varphi)$ approach

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[1] C. S. Lai, Am. J. Phys. **49**, 841 (1981).

[2] U. Backhaus and K. Schafer, Am. J. Phys. **54**, 279 (1986).

[3] M. Ribaric and L. Sustersic, *Conservation Laws and Open Questions of Classical Electrodynamics* (World Scientific Pub. Co., Singapore, 1990).

[4] In both classical and quantum theory the two are related by $\mathbf{E} = -\nabla\varphi - \partial\mathbf{A}/\partial t$, $\mathbf{B} = \nabla\times\mathbf{A}$.

is to be preferred in many cases, even in classical EM theory, because of its transparency in application.

By virtue of the freedom in EM theory to choose a gauge (gauge invariance), when employing the $(\mathbf{A},\varphi)$ potentials it is convenient to choose the Lorentz gauge

$$\nabla \cdot \mathbf{A} = -\frac{1}{c^2}\frac{\partial \varphi}{\partial t}. \qquad (2)$$

This results in simplified equations fully equivalent to Maxwell's equations for **E** and **B,**

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} = -\frac{\rho}{\varepsilon_0},\;\; \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{j}, \qquad (3)$$

in which the scalar potential φ is determined by the charge density ρ alone and the vector potential is determined by the current density $\mathbf{j}$ alone. Key to the development here it is the dependence on separate source terms for the two potentials that contributes to an independence that leads to transparency in application (see below). The solutions to Eqns. (3) are given by the retarded Green's functions

$$\varphi(\mathbf{r},t) = \int \frac{\rho\left(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c\right)}{4\pi\varepsilon_0|\mathbf{r}-\mathbf{r}'|}dV',\;\; \mathbf{A}(\mathbf{r},t) = \int \frac{\mu_0\mathbf{j}\left(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c\right)}{4\pi|\mathbf{r}-\mathbf{r}'|}dV'. \qquad (4)$$

By use of these equations and the definitions provided in Footnote 4 the usual Maxwell equations in terms of **E** and **B** as driven by charge and current densities can be rederived.

It is at this juncture that our approach differs substantially from the usual approach concerning the definitions of EM energy density and power flux, and that as a consequence provides for transparency in application. In place of the standard definition for EM energy density, (1.a), following R&S we define an EM energy density by

$$u(\mathbf{r},t) = u_A - u_\varphi + \rho\varphi, \qquad (5)$$

where u_A is an energy density defined in terms of gradients of the vector potential only,

$$u_A(\mathbf{r},t) = \frac{1}{2\mu_0}\sum_i\left[\frac{1}{c^2}\left(\frac{\partial A_i}{\partial t}\right)^2 + |\nabla A_i|^2\right], \qquad (6)$$

and u_φ is an energy density defined in terms of gradients of the scalar potential only,

$$u_\varphi(\mathbf{r},t) = \frac{1}{2}\varepsilon_0\left[\frac{1}{c^2}\left(\frac{\partial\varphi}{\partial t}\right)^2 + |\nabla\varphi|^2\right]. \qquad (7)$$

In place of the standard definition for EM power flux (1.b), we define an EM power flux by

$$\mathbf{S}(\mathbf{r},t)=\mathbf{S}_A-\mathbf{S}_\varphi+\varphi\mathbf{j}\,, \tag{8}$$

with definitions in terms of their respective (and separate) potential gradients as well,

$$\mathbf{S}_A(\mathbf{r},t)=-\varepsilon_0 c^2\sum_i\left(\frac{\partial A_i}{\partial t}\right)\nabla A_i\;, \tag{9}$$

$$\mathbf{S}_\varphi(\mathbf{r},t)=-\varepsilon_0\left(\frac{\partial\varphi}{\partial t}\right)\nabla\varphi\,. \tag{10}$$

The associated Lorentz power density is given by an expression that parallels that based on densities defined in terms of electric and magnetic fields $(\mathbf{E},\mathbf{B})$,

$$p_L=-\frac{\partial u}{\partial t}-\nabla\cdot\mathbf{S}\,. \tag{11}$$

Finally, it can be shown that the structure outlined above can be derived from a Lagrangian density

$$L(\mathbf{r},t)=u_A(\mathbf{r},t)+\mathbf{j}\cdot\mathbf{A}-u_\varphi-\rho\varphi\,. \tag{12}$$

With regard to applications we first note by Eqns. (9) and (10) that power flux (and associated momentum transfer) depend on time derivatives $(\partial A_i/\partial t)$ and $(\partial\varphi/\partial t)$ and therefore do not attribute momentum transfer to static field distributions. This is in contrast to the definition of momentum transfer in the standard $(\mathbf{E},\mathbf{B})$ formulation where power flux (at a point) is defined in terms of a crossed-field Poynting vector product $\mathbf{S}(\mathbf{r},t)=\varepsilon_0 c^2(\mathbf{E}\times\mathbf{B})$. The latter definition leads to a possible (mistaken) inference that momentum transfer associated with the power flux can be associated with crossed static **E** and **B** fields even though there are no observable consequences of such (and, worse, the drawing of faulty conclusions that such momentum transfer can lead to, say, propulsive mechanisms). Though once fully integrated over surfaces the two approaches, $(\mathbf{A},\varphi)$ and $(\mathbf{E},\mathbf{B})$, lead to identical results, it is the point-by-point distributions that differ, with the $(\mathbf{A},\varphi)$ approach being more compatible with our intuition concerning the relationship between causal charge/current sources and field effects.

Secondly, use of the standard $(\mathbf{E},\mathbf{B})$ Poynting vector approach, as pointed out by Feynman, leads to "… a peculiar thing: when we are slowly charging a capacitor, the energy is not coming down the wires; it is coming in through the edges of the gap," a seemingly nonsensical result in his opinion.[5] Use of the $(\mathbf{A},\varphi)$ definitions for energy transfer yields instead a result in keeping with our intuition that the energy transfer is supplied by the wires.[6]

Third, when one solves for the static field distribution in the case of a (near-infinite-length) solenoid, it is found that essentially all of the magnetic flux is confined to the interior of the solenoid, none outside. As a result, the standard calculation for the energy density $u(\mathbf{r},t)=(1/2)\varepsilon_0 c^2 |\mathbf{B}|^2$ confines the magnetostatic energy distribution entirely within the solenoid, none outside, and thus (correctly) that there are no magnetic effects to be detected by classical charge motion outside either. From a quantum viewpoint, however, this seems somewhat questionable since it is known that, despite this inability to detect classical charge effects exterior to the solenoid, at the quantum level quantum interference effects of the vector potential ***A*** exterior to the solenoid can be detected (Aharonov-Bohm effect).[7] However, application of Eq. (6) in this case reveals that, despite the absence of a Lorentz force $\mathbf{F}=q\mathbf{v}\times\mathbf{B}$ on a classical charge q in motion exterior to the solenoid, half of the magnetostatic energy as defined in (6) resides in the exterior region, an intuitively appealing result when one considers that the region exterior to the solenoid does register effects due to the **B**-field flux confined within the solenoid, at least at the quantum level.[8]

In summary, we see that in the application of electromagnetic principles there has over time been a development of various alternatives with regard to definitions involving the distribution of energy density and momentum transfer by EM fields defined in terms of the variables $(\mathbf{E},\mathbf{B},\mathbf{A},\varphi)$. Since all of the various (viable) options lead to identical predictions and outcomes with regard to net integrated energy and power, from a mathematical viewpoint they are identical with regard to results. Therefore, strictly speaking, it is a matter of *aesthetic choice* as to which of the various approaches are used. Nonetheless, given the vagaries of misinterpretation that can occur in application, it appears that the approach outlined herein has much to offer.

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[5] R. Feynman, R. Leighton and M. Sands, *The Feynman Lectures on Physics, Vol. II* (Addison-Wesley Pub. Co., Menlo Park, CA, 1963), paragraph 27-5 and Fig. 27-3.

[6] R&S, p. 137 ff.

[7] Vector potential ***A*** calculable from $\mathbf{B}=\nabla\times\mathbf{A}\Rightarrow\oint\mathbf{A}\bullet\mathbf{dl}=\int\mathbf{B}\bullet\mathbf{da}$

[8] R&S, p. 138 ff.