The $(\mathbf{v}\cdot\nabla)\mathbf{A}$ Convection Term applied to the Faraday Disc Generator

1. Introduction

In order to describe voltage induction purely in terms of the magnetic vector potential **A**, several authors start with the premise that all induction comes from the full time-differential of **A** that includes the partial differentiation $\frac{\partial \mathbf{A}}{\partial t}$ which applies to transformer induction plus another term that takes account of change in **A** as seen by an electron moving through a non-uniform **A** field. They take the convection term $(\mathbf{v}\cdot\nabla)\mathbf{A}$ as used in fluid dynamics as the needed addition, and they arrive at

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} - (\mathbf{v}\cdot\nabla\mathbf{A}) \qquad (1)$$

as the general formula for the electric field **E**, produced by the three terms on the RHS, respectively the gradient of the scalar potential ϕ, transformer induction and motional induction. Then they use vector math to arrive at a different expression for $(\mathbf{v}\cdot\nabla)\mathbf{A}$

$$(\mathbf{v}\cdot\nabla\mathbf{A}) = \nabla_A(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\times\mathbf{B}) \qquad (2)$$

where the first term on the RHS is the gradient of the scalar product $\mathbf{v}\cdot\mathbf{A}$ but with all the velocity derivatives removed (hence the subscript $_A$) and the second term is the well known motional induction from movement through a magnetic **B** field as taught by Fleming's RH and LH rules. (The appendix shows this derivation in detail.) Thus their new expression for **E** is

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\times\mathbf{B}) - \nabla_A(\mathbf{v}\cdot\mathbf{A}) \qquad (3)$$

in which the first three terms are found in any text dealing with EM theory and the fourth term is a new addition.

In this paper when we take the convection term $(\mathbf{v}\cdot\nabla)\mathbf{A}$ motional induction and apply this to the Faraday disc homopolar generator we find that it does not yield the actual voltage produced, hence we challenge the formula (1) and (3) as being definitive. We look into why the discrepancy arises and arrive at a different perspective for a definitive expression for **E**.

2. Homopolar Induction.

In cylindrical coordinates the vector $-\mathbf{E} = (\mathbf{v}\cdot\nabla)\mathbf{A}$ is given by

$$\begin{aligned}-\mathbf{E} = {}& \mathbf{a}_r\left[v_r\frac{\partial A_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial A_r}{\partial \theta} - \frac{v_\theta A_\theta}{r} + v_z\frac{\partial A_r}{\partial z}\right] \\ & + \mathbf{a}_\theta\left[v_r\frac{\partial A_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{v_\theta A_r}{r} + v_z\frac{\partial A_\theta}{\partial z}\right] \\ & + \mathbf{a}_z\left[v_r\frac{\partial A_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial A_z}{\partial \theta} + v_z\frac{\partial A_z}{\partial z}\right]\end{aligned} \qquad (4)$$

where $\mathbf{a}_r$, $\mathbf{a}_\theta$ and $\mathbf{a}_z$ are the unit vectors. In the Faraday homopolar generator we have a disc rotating about the z axis where v_z =0. The only induction of interest is the radial term, hence we only need consider the $\mathbf{a}_r$ terms in (4)

$$-E_r = \mathbf{a}_r\left[v_r\frac{\partial A_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial A_r}{\partial \theta} - \frac{v_\theta A_\theta}{r} + v_z\frac{\partial A_r}{\partial z}\right] \qquad (5)$$

The **A** field forms concentric circles about the z axis hence both A_r and A_z are also zero, so this reduces to

$$-E_r = \mathbf{a}_r\left[-\frac{v_\theta A_\theta}{r}\right] \tag{6}$$

We know that

$$v_\theta = \omega r \tag{7}$$

where ω is the angular velocity and we also know that A_θ rises from zero at the center to a maximum value at the outer rim of the magnet, and its value at any radius r is given by

$$A_\theta = \frac{Br}{2} \tag{8}$$

[This follows from the fact that the closed line integral of the **A** field is equal to the flux enclosed, hence with circular closures at radius r we obtain $2\pi r A_\theta = \pi r^2 B$]

Thus the induced radial **E** field as given by (6), (7) and (8) is

$$E_r = \frac{\omega Br}{2} \tag{9}$$

When integrated from center to rim at radius R we get the voltage

$$V_{homopolar} = \frac{\omega BR^2}{4} \tag{10}$$

The expected **E** = **v** X **B** induction when integrated from center to rim R gives rise to the well known homopolar voltage

$$V_{homopolar} = \frac{\omega BR^2}{2} \tag{11}$$

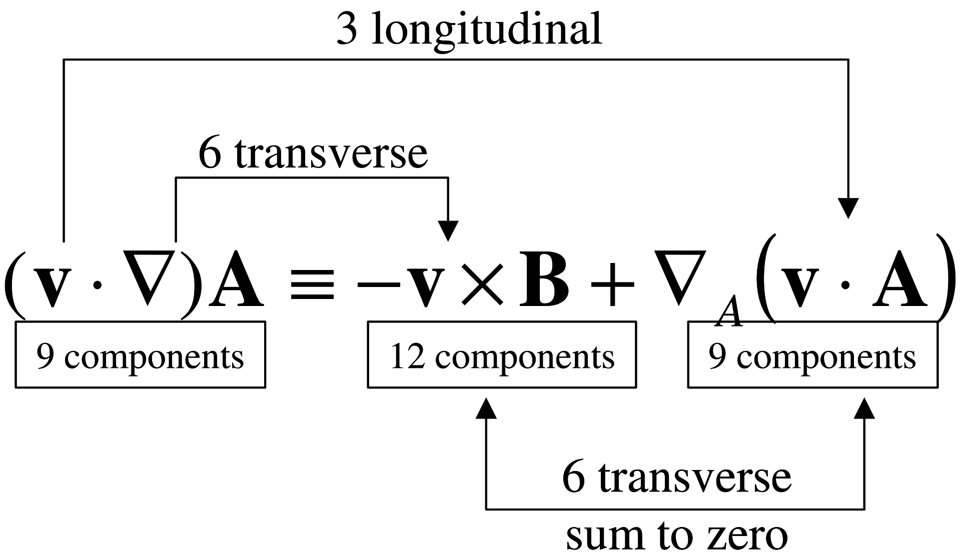
Thus there is a discrepancy between the two methods!

3. A Revised Perspective

If we look again at the identity (2)

$$(\mathbf{v}\cdot\nabla)\mathbf{A} \equiv -\mathbf{v}\times\mathbf{B} + \nabla_A(\mathbf{v}\cdot\mathbf{A}) \tag{2}$$

there is a subtlety in this equivalence that is not generally appreciated. The convective term on the LH side has 9 components, 3 longitudinal and 6 transverse, see (A1) in the appendix. The first term on the RH side of (2) has 12 components all of which are transverse, see (A2), but only 6 of these are inherited from the LH side. The last term on the RH side has 9 components, 3 longitudinal that are inherited from the LH side and 6 transverse that account for the “missing” 6. The situation is summed up in the following figure.



Note that the 6 transverse components shared by the two RH side terms sum to zero, as they must do for the equivalence to hold. If the LH side is the true defining formula then any calculations using only the $\mathbf{v}\times\mathbf{B}$ term on the RH side must give the wrong result since it includes components that would be negated by the last term. This is easily demonstrated in the Faraday disc homopolar generator where the convective $(\mathbf{v}\cdot\nabla)\mathbf{A}$ formula (or its full equivalent (2) discussed here) yields half the output voltage as calculated using just the $\mathbf{v}\times\mathbf{B}$ motional induction. That $\mathbf{v}\times\mathbf{B}$ motional induction has been in use for so long that it beggars belief that it consistently yields incorrect results, hence this calls into question the whole validity of (2).

If we transpose (2) we create the equivalence

$$\mathbf{v}\times\mathbf{B} \equiv \nabla_A(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\nabla)\mathbf{A} \qquad (12)$$

which yields the following situation.

$$\boxed{\text{12 components}}\ \mathbf{v}\times\mathbf{B} \equiv \nabla_A(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\nabla)\mathbf{A}$$

3 longitudinal sum to zero

Motional induction $\mathbf{v}\times\mathbf{B}$ obtains its 12 transverse components from the two vector identities on the RH side, but there the three longitudinal components sum to zero. This explains why the convective $(\mathbf{v}\cdot\nabla)\mathbf{A}$ formula yields half the output voltage as calculated using just the $\mathbf{v}\times\mathbf{B}$ motional induction, it only contains half the terms. It appears that the vector math that achieves (2) is not the correct way of doing things. Examination of the terms in $\mathbf{v}\times\mathbf{B}$ and comparing them with terms in $\nabla\times\mathbf{A}$ (which of course gives **B)** suggests that the convective $(\mathbf{v}\cdot\nabla)\mathbf{A}$ formula only applies to **A** fields where $\mathbf{B}=0$, and is invalid when **B** is present.

If we take the case where $\mathbf{B}=0$ we get from (12)

$$(\mathbf{v}\cdot\nabla)\mathbf{A} = \nabla_A(\mathbf{v}\cdot\mathbf{A}) \qquad (13)$$

That $\mathbf{B}=0$ condition where $\nabla\times\mathbf{A}$ (curl **A**) is zero causes the transverse components of each side of (13) to be equal in value, and they both have identical longitudinal components. Thus it is quite possible that either is valid in this situation and it doesn't matter which one we chose. We therefore offer the definitive **E** field condition as

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\times\mathbf{B}) \text{ for } \mathbf{A} \text{ fields that have curl}$$

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} - \nabla_A(\mathbf{v}\cdot\mathbf{A}) \text{ for } \mathbf{A} \text{ fields that do not have curl} \qquad (14)$$

$$\text{or } \mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t} - (\mathbf{v}\cdot\nabla\mathbf{A}) \text{ for } \mathbf{A} \text{ fields that do not have curl}$$

There is some evidence from experiments with the Marinov slip-ring generator and the Distinti Paradox 2 that longitudinal induction (along the velocity direction) as predicted by $(\mathbf{v}\cdot\nabla)\mathbf{A}$ or $\nabla_A(\mathbf{v}\cdot\mathbf{A})$ is possible.

Appendix.

The Cartesian components of $(\mathbf{v}\cdot\nabla)\mathbf{A}$ are:

$$
(\mathbf{v}\cdot\nabla)\mathbf{A} = \begin{array}{l}
\mathbf{a}_x\left[\overline{v_x\frac{\partial A_x}{\partial x}} + \boxed{v_y\frac{\partial A_x}{\partial y} + v_z\frac{\partial A_x}{\partial z}}\right] \\
\mathbf{a}_y\left[\boxed{v_x\frac{\partial A_y}{\partial x}} + \overline{v_y\frac{\partial A_y}{\partial y}} + \boxed{v_z\frac{\partial A_y}{\partial z}}\right] \\
\mathbf{a}_z\left[\boxed{v_x\frac{\partial A_z}{\partial x} + v_y\frac{\partial A_z}{\partial y}} + \overline{v_z\frac{\partial A_z}{\partial z}}\right]
\end{array} \qquad \text{(A1)}
$$

where the overbars denote the longitudinal components parallel to the velocity and the coloured rectangles denote the transverse components.

The Cartesian components of $\mathbf{v}\times\mathbf{B}$ are:

$$
\mathbf{v}\times\mathbf{B} = \begin{array}{l}
\mathbf{a}_x\left[v_y\left(\frac{\partial A_y}{\partial x} - \boxed{\frac{\partial A_x}{\partial y}}\right) - v_z\left(\boxed{\frac{\partial A_x}{\partial z}} - \frac{\partial A_z}{\partial x}\right)\right] \\
\mathbf{a}_y\left[v_z\left(\frac{\partial A_z}{\partial y} - \boxed{\frac{\partial A_y}{\partial z}}\right) - v_x\left(\boxed{\frac{\partial A_y}{\partial x}} - \frac{\partial A_x}{\partial y}\right)\right] \\
\mathbf{a}_z\left[v_x\left(\frac{\partial A_x}{\partial z} - \boxed{\frac{\partial A_z}{\partial x}}\right) - v_y\left(\boxed{\frac{\partial A_z}{\partial y}} - \frac{\partial A_y}{\partial z}\right)\right]
\end{array} \qquad \text{(A2)}
$$

where there are no longitudinal components. It is seen that only half the transverse components (as indicated by the coloured rectangles) get carried over from (A1)

The Cartesian components of $\nabla(\mathbf{v}\cdot\mathbf{A})$ are:

$$
\nabla(\mathbf{v}\cdot\mathbf{A}) = \begin{array}{l}
\mathbf{a}_x\left[v_x\frac{\partial A_x}{\partial x} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x} + A_x\frac{\partial v_x}{\partial x} + A_y\frac{\partial v_y}{\partial x} + A_z\frac{\partial v_z}{\partial x}\right] \\
\mathbf{a}_y\left[v_x\frac{\partial A_x}{\partial y} + v_y\frac{\partial A_y}{\partial y} + v_z\frac{\partial A_z}{\partial y} + A_x\frac{\partial v_x}{\partial y} + A_y\frac{\partial v_y}{\partial y} + A_z\frac{\partial v_z}{\partial y}\right] \\
\mathbf{a}_z\left[v_x\frac{\partial A_x}{\partial z} + v_y\frac{\partial A_y}{\partial z} + v_z\frac{\partial A_z}{\partial z} + A_x\frac{\partial v_x}{\partial z} + A_y\frac{\partial v_y}{\partial z} + A_z\frac{\partial v_z}{\partial z}\right]
\end{array} \qquad \text{(A3)}
$$

which include spatial derivatives of velocity. We are only interested in time variations of **A** since it is only these that induce an **E** field. If we denote (A3) with the derivatives of velocity suppressed as $\nabla_A(\mathbf{v}\cdot\mathbf{A})$ then its components are:

$$
\nabla_A(\mathbf{v}\cdot\mathbf{A}) = \begin{array}{l}
\mathbf{a}_x\left[\overline{v_x\frac{\partial A_x}{\partial x}} + v_y\frac{\partial A_y}{\partial x} + v_z\frac{\partial A_z}{\partial x}\right] \\
\mathbf{a}_y\left[v_x\frac{\partial A_x}{\partial y} + \overline{v_y\frac{\partial A_y}{\partial y}} + v_z\frac{\partial A_z}{\partial y}\right] \\
\mathbf{a}_z\left[v_x\frac{\partial A_x}{\partial z} + v_y\frac{\partial A_y}{\partial z} + \overline{v_z\frac{\partial A_z}{\partial z}}\right]
\end{array} \qquad \text{(A4)}
$$

where the overbars represent the longitudinal components carried over from (A1). The remaining terms account for the transverse components in (A2) that were not carried over from (A1)