The $(\mathbf{v}\cdot\nabla)\mathbf{A}$ Convection Term applied to the Faraday Disc Generator

In cylindrical coordinates the vector $-\mathbf{E}=(\mathbf{v}\cdot\nabla)\mathbf{A}$ is given by

$$-\mathbf{E}=\mathbf{a}_r\left[v_r\frac{\partial A_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial A_r}{\partial\theta}-\frac{v_\theta A_\theta}{r}+v_z\frac{\partial A_r}{\partial z}\right]$$
$$+\mathbf{a}_\theta\left[v_r\frac{\partial A_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial A_\theta}{\partial\theta}+\frac{v_\theta A_r}{r}+v_z\frac{\partial A_\theta}{\partial z}\right] \qquad (1)$$
$$+\mathbf{a}_z\left[v_r\frac{\partial A_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial A_z}{\partial\theta}+v_z\frac{\partial A_z}{\partial z}\right]$$

where $\mathbf{a}_r$, $\mathbf{a}_\theta$ and $\mathbf{a}_z$ are the unit vectors. In the Faraday homopolar generator we have a disc rotating about the z axis where $v_z=0$. The only induction of interest is the radial term, hence we only need consider the $\mathbf{a}_r$ terms in (1)

$$-E_r=\mathbf{a}_r\left[v_r\frac{\partial A_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial A_r}{\partial\theta}-\frac{v_\theta A_\theta}{r}+v_z\frac{\partial A_r}{\partial z}\right] \qquad (2)$$

The **A** field forms concentric circles about the z axis hence both A_r and A_z are also zero, so this reduces to

$$-E_r=\mathbf{a}_r\left[-\frac{v_\theta A_\theta}{r}\right] \qquad (3)$$

We know that

$$v_\theta=\omega r \qquad (4)$$

where ω is the angular velocity and we also know that A_θ rises from zero at the center to a maximum value at the outer rim of the magnet, and its value at any radius r is given by

$$A_\theta=\frac{Br}{2} \qquad (5)$$

[This follows from the fact that the closed line integral of the A field is equal to the flux enclosed, hence with circular closures at radius r we obtain $2\pi rA_\theta=\pi r^2B$]

Thus the induced radial **E** field as given by (3), (4) and (5) is

$$E_r=\frac{\omega Br}{2} \qquad (6)$$

When integrated from center to rim R we get the voltage

$$V_{homopolar}=\frac{\omega BR^2}{4} \qquad (7)$$

The expected $\mathbf{E}=\mathbf{v}\times\mathbf{B}$ induction when integrated from center to rim R gives rise to the well known homopolar voltage

$$V_{homopolar}=\frac{\omega BR^2}{2} \qquad (8)$$

Thus there is a discrepancy between the two methods!.

Turning to the Faraday disc homopolar motor, we are interested in the force in the θ direction due to charge movement in the r direction. Thus only the $\mathbf{a}_\theta$ terms in (1) are applicable

$$-E_\theta = \mathbf{a}_\theta\left[v_r\frac{\partial A_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial A_\theta}{\partial \theta}+\frac{v_\theta A_r}{r}+v_z\frac{\partial A_\theta}{\partial z}\right] \tag{9}$$

Since A_r=0, v_z=0 and $\frac{\partial A_\theta}{\partial \theta}=0$ this reduces to

$$-E_\theta = \mathbf{a}_\theta\left[v_r\frac{\partial A_\theta}{\partial r}\right] \tag{10}$$

From (5) we get $\frac{\partial A_\theta}{\partial r}=\frac{B}{2}$ hence (10) becomes

$$E_\theta = \frac{-v_r B}{2} \tag{11}$$

Whereas from **E** = **v** X **B** the induction is

$$E_\theta = +v_r B \tag{12}$$

which has twice the magnitude and is of opposite polarity to (11).

Clearly something is wrong when the basic equation from which the supplementary one is derived does not give results that agree with the supplementary one. Do we need to adjust the components in (1) by using partial derivatives of (unit) vectors as describes in Boast where $\frac{\partial \mathbf{a}_r}{\partial \theta}=\mathbf{a}_\theta$ and $\frac{\partial \mathbf{a}_\theta}{\partial \theta}=-\mathbf{a}_r$? Let's try it.

$$-\mathbf{E} = \mathbf{a}_r\left[v_r\frac{\partial A_r}{\partial r}+\frac{v_\theta A_\theta}{r}-\frac{v_\theta A_\theta}{r}+v_z\frac{\partial A_r}{\partial z}\right]$$

$$+\mathbf{a}_\theta\left[v_r\frac{\partial A_\theta}{\partial r}-\frac{v_\theta A_\theta}{r}+\frac{v_\theta A_r}{r}+v_z\frac{\partial A_\theta}{\partial z}\right]$$

$$+\mathbf{a}_z\left[v_r\frac{\partial A_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial A_z}{\partial \theta}+v_z\frac{\partial A_z}{\partial z}\right]$$

Removing all the zero terms leaves

$$-\mathbf{E} = \mathbf{a}_r\left[v_r\frac{\partial A_r}{\partial r}\right]$$

$$+\mathbf{a}_\theta\left[v_r\frac{\partial A_\theta}{\partial r}\right]$$

Now the v_θ terms have disappeared and that can't be right.

So let's try doing it with rectangular coordinates. From (5) the circular **A** field is given by

$$A_y = \frac{Bx}{2}$$

$$A_x = -\frac{By}{2}$$

The terms of interest are

$$(\mathbf{v}\cdot\nabla\mathbf{A}) = \mathbf{a}_x\left[v_x\,\frac{\partial A_x}{\partial x} + v_y\,\frac{\partial A_x}{\partial y}\right]$$
$$+\,\mathbf{a}_y\left[v_x\,\frac{\partial A_y}{\partial x} + v_y\,\frac{\partial A_y}{\partial y}\right]$$

Taking the radial velocity along the x axis where y = 0 and therefore $A_x = 0$ we get simply $E_y = -v_x\,\frac{\partial A_y}{\partial x} = \frac{-v_x B}{2}$ which is still out by a factor of 2 (the negative sign can be accounted for because we have used a CCW field and not a CW one.) Using the convection formula gives the wrong answer for the homopolar machine so why should it give the correct answer for the slip ring machine?

$$\mathbf{v}\times\mathbf{B} = \mathbf{a}_r(v_\theta B_z - v_z B_\theta)$$
$$+\,\mathbf{a}_\theta(v_z B_r - v_r B_z)$$
$$+\,\mathbf{a}_z(v_r B_\theta - v_\theta B_r)$$

$$\mathbf{B} = \nabla\times\mathbf{A} = \mathbf{a}_r\left(\frac{1}{r}\frac{\partial A_z}{\partial\theta} - \frac{\partial A_\theta}{\partial z}\right)$$
$$+\,\mathbf{a}_\theta\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)$$
$$+\,\mathbf{a}_z\left(\frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)$$

With the homopolar A field forming circles in the rθ plane $A_z = 0$ and $\frac{\partial A}{\partial z} = 0$ this reduces to

$$\mathbf{B} = \nabla\times\mathbf{A} = \mathbf{a}_z\left(\frac{\partial A_\theta}{\partial r} - \frac{1}{r}\frac{\partial A_r}{\partial\theta}\right)$$

but if $A_r = 0$ and $\frac{\partial A_r}{\partial\theta} = 0$ we lose the last term and get $\mathbf{B} = \mathbf{a}_z\left(\frac{\partial A_\theta}{\partial r}\right)$ which is out by a factor of 2. The only way we can correct this is to put $\frac{\partial A_r}{\partial\theta} = -A_\theta$ then the second term becomes $-\frac{A_\theta}{r}$ and that puts things right.

$$\mathbf{v}\times\mathbf{B} = \mathbf{a}_r\left(v_\theta\,\frac{\partial A_\theta}{\partial r} - \frac{v_\theta}{r}\frac{\partial A_r}{\partial\theta} - v_z\,\frac{\partial A_r}{\partial z} + v_z\,\frac{\partial A_z}{\partial r}\right)$$
$$+\,\mathbf{a}_\theta\left(\frac{v_z}{r}\frac{\partial A_z}{\partial\theta} - v_z\,\frac{\partial A_\theta}{\partial z} - v_r\,\frac{\partial A_\theta}{\partial r} + \frac{v_r}{r}\frac{\partial A_r}{\partial\theta}\right)$$
$$+\,\mathbf{a}_z\left(v_r\,\frac{\partial A_r}{\partial z} - v_r\,\frac{\partial A_z}{\partial r} - \frac{v_\theta}{r}\frac{\partial A_z}{\partial\theta} + v_\theta\,\frac{\partial A_\theta}{\partial z}\right)$$

With $A_z = 0$, $v_z = 0$ and $\frac{\partial A}{\partial z} = 0$ this simplifies to

$$\mathbf{v}\times\mathbf{B}=\mathbf{a}_r\left(v_\theta\frac{\partial A_\theta}{\partial r}-\frac{v_\theta}{r}\frac{\partial A_r}{\partial\theta}\right)$$
$$+\mathbf{a}_\theta\left(-v_r\frac{\partial A_\theta}{\partial r}+\frac{v_r}{r}\frac{\partial A_r}{\partial\theta}\right)$$

and with the substitution $\frac{\partial A_r}{\partial\theta}=-A_\theta$ we get

$$\mathbf{v}\times\mathbf{B}=\mathbf{a}_r\left(v_\theta\frac{\partial A_\theta}{\partial r}+\frac{v_\theta A_\theta}{r}\right)$$
$$+\mathbf{a}_\theta\left(-v_r\frac{\partial A_\theta}{\partial r}-\frac{v_r A_\theta}{r}\right)$$

which now works for both homopolar motor and generator.

Now look at the convective formula

$$-\mathbf{E}=\mathbf{a}_r\left[v_r\frac{\partial A_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial A_r}{\partial\theta}-\frac{v_\theta A_\theta}{r}+v_z\frac{\partial A_r}{\partial z}\right]$$
$$+\mathbf{a}_\theta\left[v_r\frac{\partial A_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial A_\theta}{\partial\theta}+\frac{v_\theta A_r}{r}+v_z\frac{\partial A_\theta}{\partial z}\right]$$
$$+\mathbf{a}_z\left[v_r\frac{\partial A_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial A_z}{\partial\theta}+v_z\frac{\partial A_z}{\partial z}\right]$$

with the substitution $\frac{\partial A_r}{\partial\theta}=-A_\theta$ and from Boast's description of vector differentiation we surmise $\frac{\partial A_\theta}{\partial\theta}=A_r$ we then get

$$-\mathbf{E}=\mathbf{a}_r\left[v_r\frac{\partial A_r}{\partial r}-2\frac{v_\theta A_\theta}{r}+v_z\frac{\partial A_r}{\partial z}\right]$$
$$+\mathbf{a}_\theta\left[v_r\frac{\partial A_\theta}{\partial r}+2\frac{v_\theta A_r}{r}+v_z\frac{\partial A_\theta}{\partial z}\right]$$
$$+\mathbf{a}_z\left[v_r\frac{\partial A_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial A_z}{\partial\theta}+v_z\frac{\partial A_z}{\partial z}\right]$$

Let's compare the $\mathbf{v} \times \mathbf{B}$ with the $-(\mathbf{v} \cdot \nabla)\mathbf{A}$

$$\mathbf{v} \times \mathbf{B} = \mathbf{a}_r \left(v_\theta \frac{\partial A_\theta}{\partial r} - \boxed{\frac{v_\theta}{r} \frac{\partial A_r}{\partial \theta}} - \boxed{v_z \frac{\partial A_r}{\partial z}} + v_z \frac{\partial A_z}{\partial r} \right)$$
$$+ \mathbf{a}_\theta \left(\frac{v_z}{r} \frac{\partial A_z}{\partial \theta} - \boxed{v_z \frac{\partial A_\theta}{\partial z}} - \boxed{v_r \frac{\partial A_\theta}{\partial r}} + \frac{v_r}{r} \frac{\partial A_r}{\partial \theta} \right)$$
$$+ \mathbf{a}_z \left(v_r \frac{\partial A_r}{\partial z} - \boxed{v_r \frac{\partial A_z}{\partial r}} - \boxed{\frac{v_\theta}{r} \frac{\partial A_z}{\partial \theta}} + v_\theta \frac{\partial A_\theta}{\partial z} \right)$$

$$-(\mathbf{v} \cdot \nabla \mathbf{A}) = \mathbf{a}_r \left[-v_r \frac{\partial A_r}{\partial r} - \boxed{\frac{v_\theta}{r} \frac{\partial A_r}{\partial \theta}} + \frac{v_\theta A_\theta}{r} - \boxed{v_z \frac{\partial A_r}{\partial z}} \right]$$
$$+ \mathbf{a}_\theta \left[-\boxed{v_r \frac{\partial A_\theta}{\partial r}} - \frac{v_\theta}{r} \frac{\partial A_\theta}{\partial \theta} - \frac{v_\theta A_r}{r} - \boxed{v_z \frac{\partial A_\theta}{\partial z}} \right]$$
$$+ \mathbf{a}_z \left[-\boxed{v_r \frac{\partial A_z}{\partial r}} - \boxed{\frac{v_\theta}{r} \frac{\partial A_z}{\partial \theta}} - v_z \frac{\partial A_z}{\partial z} \right]$$

Common terms

Now with v_z and $A_z = 0$ and $\frac{\partial A}{\partial z} = 0$

$$\mathbf{v} \times \mathbf{B} = \mathbf{a}_r \left(v_\theta \frac{\partial A_\theta}{\partial r} - \boxed{\frac{v_\theta}{r} \frac{\partial A_r}{\partial \theta}} \right)$$
$$+ \mathbf{a}_\theta \left(-\boxed{v_r \frac{\partial A_\theta}{\partial r}} + \frac{v_r}{r} \frac{\partial A_r}{\partial \theta} \right)$$

$$-(\mathbf{v} \cdot \nabla \mathbf{A}) = \mathbf{a}_r \left[-v_r \frac{\partial A_r}{\partial r} - \boxed{\frac{v_\theta}{r} \frac{\partial A_r}{\partial \theta}} + \frac{v_\theta A_\theta}{r} \right]$$
$$+ \mathbf{a}_\theta \left[-\boxed{v_r \frac{\partial A_\theta}{\partial r}} - \frac{v_\theta}{r} \frac{\partial A_\theta}{\partial \theta} - \frac{v_\theta A_r}{r} \right]$$