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Parametric Up-conversion by the Use of Non-linear Resistance and Capacitance

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Summary: The power gain of parametric up-converters using both non-linear resistance and capacitance is found for the condition of conjugate matching at the output termination only. It is shown that for the elements pumped in quadrature it is possible to obtain input impedances with a negative real part, and that under these conditions the gain may be arbitrarily large in principle for any value of pump frequency greater than zero. It is also shown that the device may be used as a negative-resistance parametric amplifier in which the pump frequency is lower than the signal frequency.

Previous results showing that the up-converters may be made unidirectional in operation are confirmed, and it is shown that the gain under these conditions may be optimized by suitable adjustment of the mark/space ratio of the resistance variation. Attempts are made on physical grounds to explain both the unidirectional operation and also the high gain possible with R-C converters. Finally, some experimental confirmation of the theory is given.

List of Symbols

$C(t)$	Time-varying capacitance.
$C_0, 2C_1, 2C_r$	Fourier coefficients of $C(t)$.
$\frac{1}{C}(t)$	Time-varying elastance.
$\left(\frac{1}{C}\right)_0, \left(\frac{2}{C}\right)_1, \left(\frac{2}{C}\right)_r$	Fourier coefficients of $\frac{1}{C}(t)$.
$C_0', \frac{1}{2}C_1', \frac{1}{2}C_r'$	The reciprocal of the Fourier coefficients of $\frac{1}{C}(t)$.
$g(t)$	Time-varying conductance.
$g_0, 2g_1, 2g_r$	Fourier coefficients of $g(t)$.
g_f	Forward conductance.
g_b	Reverse (or back) conductance.
G_0'	Input conductance.
G_{+1}	Load conductance.
i_0	Current at frequency ω_q.
i_{+1}	Current at frequency $\omega_p+\omega_q$.
I_0	Source current (at frequency ω_q).
I_{+1}	Source current (at frequency $\omega_p+\omega_q$).
I_0'	Equivalent source current (at frequency ω_q).
I_{+1}'	Equivalent source current (at frequency $\omega_p+\omega_q$).
n	$\sqrt{g_f/g_b}$
P_f	Forward power gain.
P_r	Reverse power gain.
$r(t)$	Time-varying resistance.
$r_0, 2r_1, 2r_r$	Fourier coefficients of $r(t)$.
R_0	Source resistance.
R_{+1}	Load resistance.
R_{+1}'	Output resistance.
s	Ratio of 'mark' to 'mark plus space'.
v_0	Voltage at frequency ω_q.
v_{+1}	Voltage at frequency $\omega_p+\omega_q$.
V_0	Source voltage (at frequency ω_q).
V_{+1}'	Equivalent source voltage (at frequency $\omega_p+\omega_q$).
Y_0	Source admittance.
Y_0'	Input admittance.
Y_{+1}	Load admittance.
Y_{+1}'	Output admittance.
Z_0	Source impedance.
Z_{+1}	Load impedance.
Z_{+1}'	Output impedance.
θ_0	Phase angle of v_0.
θ_{+1}	Phase angle of v_{+1}.
τ	Pulse width.
ω_p	Pump frequency.
ω_q	Signal frequency.
$\omega_p+\omega_q$	Output frequency.

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1. Introduction

Parametric up-converters using back-biased semiconductor diodes as non-linear capacitors are well-known,[1] and have found useful application for high-frequency, low-noise amplification. Usually such devices are of the three frequency, non-inverting type —that is, only currents or voltages at the signal frequency, the diode pump frequency, and the sum of these frequencies occur in the circuit. The output is taken at the sum frequency, so that the wanted sideband is not inverted.

The equations of Manley and Rowe[2] establish that the maximum power gain of a parametric converter such as has been discussed, in which the non-linear element is a pure capacitor, is the ratio of the output to the input frequency. Edwards[3] has shown that the addition of a non-linear resistor to such a circuit if pumped in phase with the capacitor merely results in a degradation of this performance. There have been other papers in this field[4,5,6] and in particular Engelbrecht[5] has established recently that if the non-linear resistor is pumped in quadrature with the capacitor it is possible to obtain greater power gain than with the capacitor alone, for large ratios of output to input frequency. It was also established that it is possible to make such a circuit unidirectional —that is, to have a finite gain from input to output, and zero gain in the reverse direction. This is advantageous, since under these conditions noise from a following stage cannot be transferred to the converter input. The work reported in this paper is in the main a more general analysis of the case of the resistance and capacitance pumped in quadrature than has hitherto been made. The condition of conjugate matching of source impedance and input impedance has been removed, and it is shown that without that limitation the gain of either series or parallel R,C circuits may become infinite for certain element values and pump phasing. Attempts are made to explain both the gain and directional properties of the devices in simple terms. The corresponding down-converters are briefly mentioned. No consideration is given to lower-sideband (inverting) up-converters in this paper, since it is felt that the important properties of these devices are already adequately recorded elsewhere.[5,6,11]

2. The Parametric Up-Converter with Parallel R and C Elements

The circuit to be analysed is shown in Fig. 1. It will be assumed throughout this work that the pump voltages across the resistor and capacitor are much larger than the corresponding signal voltages, so that the value of resistance or capacitance at any time is controlled entirely by the pump(s)†. That said, there is no theoretical need to consider the pump voltages further, since the circuit analysis may be undertaken from a consideration of the *time*-variation of the two elements. Hence in Fig. 1 no pump generators are shown, the capacitance and resistance being merely represented as functions of time. The band-stop filters in the circuit are considered to be ideal, so that only a voltage at signal frequency (ω_q) appears across the source admittance (Y_0), and a corresponding voltage at upper sideband frequency ($\omega_p+\omega_q$) appears across the load (Y_{+1}). The nomenclature used for currents and voltages at various frequencies is the same as that used in a previous paper[7,8] The time-varying capacitance may in general be represented by a Fourier series, i.e.

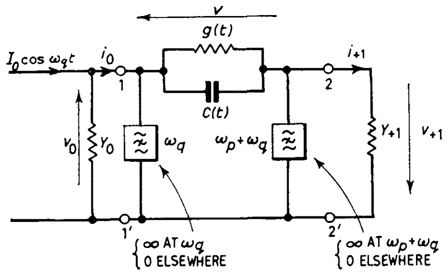

Fig. 1. Parametric converter with parallel R and C elements.

$$C(t) = C_0 + \sum_{r=1}^{\infty} 2C_r \cos r\omega_p t \qquad \ldots\ldots(1)$$

For convenience the resistor will be considered as time-varying conductance. Since this is pumped so that it is in quadrature with the capacitor, if the conductance variation is $\pi/2$ *ahead* of the capacitance,

$$g(t) = g_0 + \sum_{r=1}^{\infty} 2g_r \cos r\left(\omega_p t - \frac{\pi}{2}\right) \qquad \ldots\ldots(2)$$

$$= g_0 + 2g_1 \sin \omega_p t - 2g_2 \cos 2\omega_p t \ldots \qquad \ldots\ldots(3)$$

On the other hand, if the conductance variation is $\pi/2$ *behind* that of the capacitance,

$$g(t) = g_0 + \sum_{r=1}^{\infty} 2g_r \cos r\left(\omega_p t + \frac{\pi}{2}\right) \qquad \ldots\ldots(4)$$

$$= g_0 - 2g_1 \sin \omega_p t - 2g_2 \cos 2\omega_p t \ldots \qquad \ldots\ldots(5)$$

In our analysis we shall consider the latter case, and be able to deal with the former when necessary by considering g_1 to be negative. No higher coefficients will enter into the equations. The current through the two elements is

$$\frac{\mathrm{d}}{\mathrm{d}t}\{C(t).v\} \qquad \ldots\ldots(6)$$

and

$$g(t).v \qquad \ldots\ldots(7)$$

† It is also assumed that if there are separate pumps for the resistor and capacitor, these do not interact. In practice this will mean balanced resistor and capacitor networks.

where v is the voltage across them.

$$v = v_0\cos(\omega_q t+\theta_0)+v_{+1}\cos(\overline{\omega_p+\omega_q}\,t+\theta_{+1}) \quad \ldots\ldots(8)$$

The equations for current at signal and sideband frequencies in the circuit may now be formulated. If v_0 and v_{+1} are considered to be complex in order to include the effect of the phase angles, these are

$$I_0\cos\omega_q t = Y_0 v_0\cos\omega_q t + \left[g(t)(v_0\cos\omega_q t+v_{+1}\cos\overline{\omega_p+\omega_q}\,t)+\frac{\mathrm{d}}{\mathrm{d}t}\{C(t)(v_0\cos\omega_q t+v_{+1}\cos\overline{\omega_p+\omega_q}\,t)\}\right]_0 \quad \ldots\ldots(9)$$

and

$$0 = Y_{+1}v_{+1}\cos\overline{\omega_p+\omega_q}\,t + \left[g(t)(v_0\cos\omega_q t+v_{+1}\cos\overline{\omega_p+\omega_q}\,t)+\frac{\mathrm{d}}{\mathrm{d}t}\{C(t)(v_0\cos\omega_q t+v_{+1}\cos\overline{\omega_p+\omega_q}\,t)\}\right]_{+1} \quad \ldots\ldots(10)$$

where the suffices after the brackets indicate the frequencies at which they are to be evaluated. Evaluating the equations, replacing 'sin' by 'j cos' where necessary, the vector equations are

$$I_0 = (Y_0+g_0+\mathrm{j}\omega_q C_0)v_0+(-\mathrm{j}g_1+\mathrm{j}\omega_q C_1)v_{+1} \quad \ldots\ldots(11)$$

$$0 = (\mathrm{j}g_1+\mathrm{j}\overline{\omega_p+\omega_q}\,C_1)v_0+(Y_{+1}+g_0+\mathrm{j}\overline{\omega_p+\omega_q}\,C_0)v_{+1} \quad \ldots\ldots(12)$$

Let the terminating admittances be

$$Y_0 = G_0-\mathrm{j}\omega_q C_0 \quad \ldots\ldots(13)$$

$$Y_{+1} = G_{+1}-\mathrm{j}\overline{\omega_p+\omega_q}\,C_0 \quad \ldots\ldots(14)$$

Then the equations (11) and (12) become

$$I_0 = (g_0+G_0)v_0+\mathrm{j}(\omega_q C_1-g_1)v_{+1} \quad \ldots\ldots(15)$$

$$0 = \mathrm{j}(\overline{\omega_p+\omega_q}\,C_1+g_1)v_0+(G_{+1}+g_0)v_{+1} \quad \ldots\ldots(16)$$

The circuit to the left of the terminals 22′ may be regarded as a current generator I'_{+1} at sideband frequency in parallel with an admittance Y'_{+1} also defined at that frequency. From the circuit it is clear that

$$I'_{+1}+Y'_{+1}v_{+1} = -Y_{+1}v_{+1} \quad \ldots\ldots(17)$$

From (12), $I'_{+1}+Y'_{+1}v_{+1}$

$$= (\mathrm{j}g_1+\mathrm{j}\overline{\omega_p+\omega_q}\,C_1)v_0+(g_0+\mathrm{j}\overline{\omega_p+\omega_q}\,C_0)v_{+1} \quad \ldots(18)$$

From (15), $I'_{+1}+Y'_{+1}v_{+1}$

$$= \mathrm{j}(g_1+\overline{\omega_p+\omega_q}\,C_1)\,.\left\{\frac{I_0-\mathrm{j}(\omega_q C_1-g_1)v_{+1}}{g_0+G_0}\right\}+ +(g_0+\mathrm{j}\overline{\omega_p+\omega_q}\,C_0)v_{+1} \quad \ldots\ldots(19)$$

Therefore

$$I'_{+1}+Y'_{+1}v_{+1} = \frac{\mathrm{j}(g_1+\overline{\omega_p+\omega_q}\,C_1)}{G_0+g_0}I_0+\left\{\frac{(\overline{\omega_p+\omega_q}\,C_1+g_1)(\omega_q C_1-g_1)+(g_0+G_0)(\mathrm{j}\overline{\omega_p+\omega_q}\,C_0+g_0)}{G_0+g_0}\right\}v_{+1} \quad \ldots\ldots(20)$$

Hence
$$I'_{+1} = \frac{\mathrm{j}(g_1+\overline{\omega_p+\omega_q}\,C_1)}{G_0+g_0}I_0 \quad \ldots\ldots(21)$$

and
$$Y'_{+1} = \frac{(\overline{\omega_p+\omega_q}\,C_1+g_1)(\omega_q C_1-g_1)+g_0(g_0+G_0)}{G_0+g_0}+\mathrm{j}\overline{\omega_p+\omega_q}\,C_0 \quad \ldots\ldots(22)$$

For a maximum power output, the output admittance must be the conjugate of the load admittance.

Therefore $$(Y'_{+1})^* = Y_{+1} \quad \ldots\ldots(23)$$

From (14) and (22), it is clear that the imaginary part of Y_{+1} has been chosen correctly, and that

$$G_{+1} = g_0-\frac{(g_1-\omega_q C_1)(g_1+\overline{\omega_p+\omega_q}\,C_1)}{G_0+g_0} \quad \ldots\ldots(24)$$

The power in the load $$= \frac{|I'_{+1}|^2}{8G_{+1}} \quad \ldots\ldots(25)$$

$$= \frac{(g_1+\overline{\omega_p+\omega_q}\,C_1)^2 I_0^2}{8(G_0+g_0)\{g_0(G_0+g_0)-(g_1-\omega_q C_1)(g_1+\overline{\omega_p+\omega_q}\,C_1)\}} \quad \ldots\ldots(26)$$

The maximum available power from the source is

$$\frac{I_0^2}{8G_0} \qquad \text{......(27)}$$

so that the power gain from input to output of the circuit is

$$P_f = \frac{G_0(g_1+\overline{\omega_p+\omega_q}\,C_1)^2}{(G_0+g_0)\{g_0(G_0+g_0)-(g_1-\omega_q C_1)(g_1+\overline{\omega_p+\omega_q}\,C_1)\}} \qquad \text{......(28)}$$

P_f will be termed from henceforth the 'forward' gain. For a passive resistance element (which excludes devices which have a negative resistance over some part of their characteristic, such as the tunnel diode) $|g_1| \leqslant g_0$.

The power gain of the converter becomes infinite when

$$g_0(G_0+g_0) = (g_1-\omega_q C_1)(g_1+\overline{\omega_p+\omega_q}\,C_1) \qquad \text{......(29)}$$

For a diode for which $g_0 \simeq g_1$, this means that

$$g_0 G_0+\omega_q(\omega_p+\omega_q)C_1^2 \simeq \omega_p C_1 \,.\, g_0 \qquad \text{......(30)}$$

The equation cannot be satisfied for all values of G_0. If $G_0 = kg_0$, for instance, and $\omega_p = n\omega_q$, the resultant quadratic for g_0 will only have real roots if

$$k < \tfrac{1}{4}n^2/(n+1).$$

Therefore for small step-up ratios $(\omega_p+\omega_q)/\omega_q$ in the converter, G_0 will have to be correspondingly small to allow high gain to be obtained. From (24) it is clear that G_{+1} is always small for high gain.

If G_0 is taken to be very much less than g_0, (30) reduces to

$$g_0 \simeq \left(1+\frac{\omega_q}{\omega_p}\right).\,\omega_q C_1 \qquad \text{......(31)}$$

It can be seen from a consideration of a later equation (35) in conjunction with the condition for infinite gain that the converter so acts because the input admittance has a negative real part of $(-G_0)$. Thus the converter cannot be conjugately matched at the input end with passive terminations, and in fact shows maximum gain when the input admittance is equal to $(-Y_0)$.† For the conductance pumped $\pi/2$ rad *ahead* of the capacitance, in contrast with the case just considered, the forward gain is always finite for terminations with positive real parts. This can be seen by substituting $(-g_1)$ for g_1 in (28). Using the same approximation for g_1 as above, and if $g_0 \ll G_0 \ll \overline{\omega_p+\omega_q}\,C_1$ then the gain approaches $(\omega_p+\omega_q)/\omega_q$, so that the addition of the non-linear resistance to the circuit brings no benefits if pumped with this phase angle.

Previous work[6,11] has shown that the same conclusions about the optimum pumping angles apply when the maximum 'unconditionally-stable' power gain is being calculated; 'unconditionally-stable', that is, for any terminations at either pair of terminals 11′ and 22′.

The gain from output to input (or the 'reverse' gain) for the same terminating conditions considered for (26) may now be calculated. The circuit to the right of terminals 11′ in Fig. 1 will be regarded as a current generator (I_0') at signal frequency in parallel with an admittance Y_0'. By analogy with (21) and (22) we have that

$$I_0' = \frac{\mathrm{j}(\omega_q C_1-g_1)}{g_0+G_{+1}} I_{+1} \qquad \text{......(32)}$$

where I_{+1} is a current source of frequency $\omega_p+\omega_q$ applied to the output. Similarly

$$Y_0' = G_0'+\mathrm{j}\omega_q C_0 \qquad \text{......(33)}$$

where

$$G_0' = g_0 - \frac{(g_1-\omega_q C_1)(g_1+\overline{\omega_p+\omega_q}\,C_1)}{g_0+G_{+1}} \qquad \text{......(34)}$$

Combining (34) and (24)

$$(G_0'-g_0)(G_{+1}+g_0) = (G_{+1}-g_0)(g_0+G_0) \qquad \text{......(35)}$$

The power in the admittance Y_0 is, from (32) and (33),

$$\frac{1}{2G_0}\left\{\frac{G_0 I_0'}{G_0+G_0'}\right\}^2 \qquad \text{......(36)}$$

so that the reverse gain is, from (32) and (36),

$$P_r = \frac{4G_0 G_{+1}(\omega_q C_1-g_1)^2}{(G_0+G_0')^2(g_0+G_{+1})^2} \qquad \text{......(37)}$$

which reduces, after substitution from (35) for G_0', and (24) for G_{+1}, to

$$P_r = \frac{G_0(\omega_q C_1-g_1)^2}{(G_0+g_0)\{g_0(G_0+g_0)-(g_1-\omega_q C_1)(g_1+\overline{\omega_p+\omega_q}\,C_1)\}} \qquad \text{......(38)}$$

Thus the ratio of forward to reverse gain is given by

$$\frac{P_f}{P_r} = \left\{\frac{g_1+\overline{\omega_p+\omega_q}\,C_1}{g_1-\omega_q C_1}\right\}^2 \qquad \text{......(39)}$$

If $g_1 = \omega_q C_1$, the device is unidirectional with zero reverse gain, under which conditions, from (28) when

† In the Appendix it is shown that it is also possible to operate the device so that the *output* admittance has a negative real part, which means that the converter may be operated as a negative-resistance parametric amplifier in which the pump frequency (ω_p) is lower than the 'signal' frequency $(\omega_p+\omega_q)$.

$g_0 = G_0$ the forward gain attains a maximum value of

$$P_f = \left(\frac{g_1}{2g_0}\right)^2 \cdot \left(1 + \frac{\omega_p + \omega_q}{\omega_q}\right)^2 \quad ......(40)$$

To attain higher gain it is necessary to depart from unidirectionality. However, (31) shows that g_1 has only to be increased slightly above $\omega_q C_1$ before the gain becomes arbitrarily-high, particularly when the step-up ratio $(\omega_p+\omega_q)/\omega_q$ is large. Under these conditions, from (39) it is possible to obtain high forward gain whilst still maintaining a large ratio of forward to reverse gain. For example, if $g_1 \simeq g_0$, and $\omega_p = 3\omega_q$, the maximum unidirectional gain is, from (40), 8 dB. But if $2G_0 = g_0 \simeq 1{\cdot}9\omega_q C_1, P_f = 20{\cdot}6$ dB, $P_r = 4{\cdot}3$ dB.

The gain obtainable is therefore greater than that of a 'cold-tuned' four-frequency parametric converter using non-linear capacitance, which is $4(\omega_p+\omega_q)/\omega_q$. If such a converter is off-tuned, arbitrarily-high gain is obtainable as for the RC converter. Noise figures for both converters will be higher than for the ordinary three-frequency capacitance converter. Both noise and bandwidth properties are being investigated at present, and details will probably be published at a later date.

Note that if the converter is pumped so that the conductance waveform is $\pi/2$ rad *ahead* of that of the capacitance, when g_1 may be considered negative, the device can only be unidirectional in the opposite sense. In other words, the forward gain can be made zero whilst the reverse gain remains finite.

3. The Up-Converter with Series R and C Elements

The circuits shown in Fig. 2 and Fig. 3 are those of up-converters with series R and C elements. They are for all practical purposes identical, as shown previously,[7] and require the same equations for their analysis. A previous analysis of the circuit of Fig. 3 has been made, for conditions of conjugate matching at the input and output terminals of the converter.[5] The terminology adopted in the previous analysis differed in several ways, in particular in using the Fourier coefficients for the conductance and capacitance, rather than those of the resistance and elastance, as will be done here.

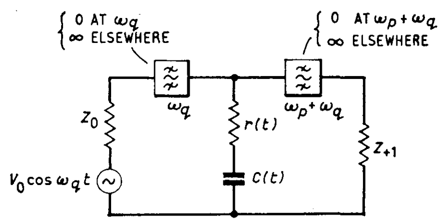

Fig. 3. Shunt form of the parametric converter with series R and C elements.

The resistance and capacitance will be considered time-varying elements in a similar manner to before, and we will consider initially that

$$\frac{1}{C}(t) = \left(\frac{1}{C_0}\right) + \sum_{r=1}^{\infty} 2\left(\frac{1}{C}\right)_r \cos r\omega_p t \quad ......(41)$$

If the resistance variation is $\pi/2$ rad *behind* this, then

$$r(t) = r_0 - 2r_1 \sin \omega_p t - 2r_2 \cos 2\omega_p t ... \quad ......(42)$$

and as before the case of the resistance variation being $\pi/2$ rad *ahead* of (41) will be dealt with by considering r_1 negative.

Using similar conventions to the previous work, if the currents through the terminations are $i_0 \cos \omega_q t$ and $i_{+1} \cos \overline{\omega_p+\omega_q t}$, then the loop voltage equations at signal and sideband frequencies are

$$V_0 \cos \omega_q t = Z_0 i_0 \cos \omega_q t + \left[r(t)(i_0 \cos \omega_q t + i_{+1} \cos \overline{\omega_p+\omega_q t}) + \frac{1}{C(t)} \int (i_0 \cos \omega_q t + i_{+1} \cos \overline{\omega_p+\omega_q t})\, dt\right]_0 \quad ......(43)$$

and

$$0 = Z_{+1} i_{+1} \cos \overline{\omega_p+\omega_q t} + \left[r(t)(i_0 \cos \omega_q t + i_{+1} \cos \overline{\omega_p+\omega_q t}) + \frac{1}{C(t)} \int (i_0 \cos \omega_q t + i_{+1} \cos \overline{\omega_p+\omega_q t})\, dt\right]_{+1} \quad ......(44)$$

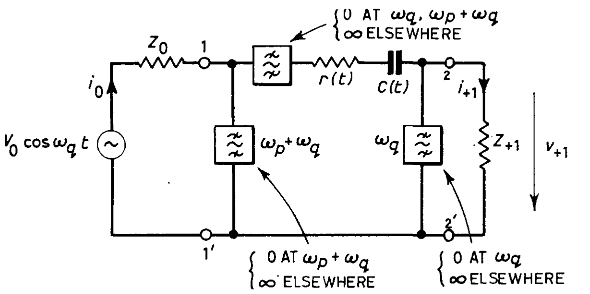

Fig. 2. Parametric converter with series R and C elements.

Evaluating the equations as before,

$$V_0 = \left(Z_0 + r_0 + \frac{1}{j\omega_q C_0'}\right) i_0 + \left(\frac{1}{j\overline{\omega_p+\omega_q} C_1'} - jr_1\right) i_{+1} \quad ......(45)$$

$$0 = \left(\frac{1}{j\omega_q C_1'} + jr_1\right) i_0 + \left(Z_{+1} + \frac{1}{j\overline{\omega_p+\omega_q} C_1'} + r_0\right) i_{+1} \quad ......(46)$$

It will be assumed that

$$Z_0 = R_0 - \frac{1}{j\omega_q C_0'} \qquad \ldots\ldots(47)$$

and

$$Z_{+1} = R_{+1} - \frac{1}{j\overline{\omega_p+\omega_q}C_1'} \qquad \ldots\ldots(48)$$

Considering the circuit to the left of terminals 2,2′ to be replaced by an equivalent voltage generator (V_{+1}') at sideband frequency in series with an impedance Z_{+1}', after some manipulation we find that

$$V_{+1}' = -\frac{j\left(r_1 - \dfrac{1}{\omega_q C_1'}\right)V_0}{Z_0 + r_0 + \dfrac{1}{j\omega_q C_0'}} \qquad \ldots\ldots(49)$$

and

$$Z_{+1}' = R_{+1}' + \frac{1}{j\overline{\omega_p+\omega_q}C_1'} \qquad \ldots\ldots(50)$$

where

$$R_{+1}' = \frac{r_0(r_0+R_0) - \left(r_1 + \dfrac{1}{\overline{\omega_p+\omega_q}C_1'}\right)\left(r_1 - \dfrac{1}{\omega_q C_1'}\right)}{r_0+R_0} \qquad \ldots\ldots(51)$$

For a conjugate match at the output terminals $R_{+1}' = R_{+1}$, so that the power in the load resistance is $|V_{+1}'|^2/8R_{+1}$. Since the maximum available power from the source is $V_0^2/8R_0$, from (49) and (51) the forward power gain may be calculated to be

$$P_f = \frac{R_0\left(r_1 - \dfrac{1}{\omega_q C_1'}\right)^2}{(r_0+R_0)\left\{r_0(r_0+R_0) - \left(r_1 + \dfrac{1}{\overline{\omega_p+\omega_q}C_1'}\right)\left(r_1 - \dfrac{1}{\omega_q C_1'}\right)\right\}} \qquad \ldots\ldots(52)$$

Since for a passive resistance element $|r_1| \leqslant r_0$, and assuming also that $\omega_p \gg \omega_q$ the denominator is approximately

$$(r_0+R_0)\left\{r_0 R_0 + \frac{1}{\omega_q(\omega_p+\omega_q)C_1'^2} + \frac{1}{\omega_q C_1'}r_1\right\} \qquad \ldots\ldots(53)$$

It is easy to see that forward gain can never become infinite. If, however, the resistance variation is now considered to be $\pi/2$ rad *ahead* of that of the elastance $1/C(t)$, the sign of r_1 is reversed, and it is clear from (53) that the gain will now become infinite when

$$r_0 R_0 + \frac{1}{\omega_q(\omega_p+\omega_q)C_1'^2} = \frac{r_1}{\omega_q C_1'} \qquad \ldots\ldots(54)$$

From (51), R_{+1} becomes very small as the gain tends to infinity.

Other calculations for this case follow the same pattern as for the parallel R-C converter and will be omitted here. It is sufficient to say that it is readily shown[5] that the device may be unidirectional with finite *forward* gain for the resistance variation $\pi/2$ *ahead* of that of $1/C(t)$, and in that case the forward gain becomes, for $r_0 = R_0$,

$$P_f = \left(\frac{r_1}{2r_0}\right)^2\left(1 + \frac{\omega_p+\omega_q}{\omega_q}\right)^2 \qquad \ldots\ldots(55)$$

4. The Optimization of the Gain of Unidirectional Converters

It has been shown (40), (55) that it is possible to make either form of up-converter unidirectional, but that if this is done the gain is not infinite, but a function of the ratio of output to input frequencies, and of g_1/g_0 or r_1/r_0. For a fixed frequency ratio, therefore, the question remains as to how to optimize the resistance or conductance ratios. In this section we shall consider how these theoretically may be made very close to unity for high-efficiency diodes. In practice it may be difficult to achieve this sort of improvement at frequencies for which the converters are most valuable.

If the diode is switched from forward to reverse conduction by a large sinusoidal voltage, then the variation of incremental resistance approximates to a square wave of equal mark/space ratio.[9] It will be assumed that this square-wave variation of diode conductance switches instantaneously from a small value of conductance (the reverse conductance, g_b) to a much larger value (the forward conductance, g_f), and vice versa. If a d.c. bias is applied in addition to the diode, then this mark/space ratio varies. It will now be shown under what conditions a square-wave variation in diode resistance optimizes the ratio of g_1/g_0, or of r_1/r_0.

It has been shown elsewhere[10] that a square-wave variation of incremental conductance may be represented by the Fourier series

$$g(t) = g_b + s(g_f - g_b) + \frac{2}{\pi}(g_f - g_b) \times \sum_{r=1}^{\infty} \frac{\sin(r\pi s)}{r}\cos r\omega_p t \qquad \ldots\ldots(56)$$

where s is the ratio of 'mark' to 'mark plus space' (see Fig. 4) and

$$n^2 = g_f/g_b \gg 1 \quad \text{for all efficient diodes.}$$

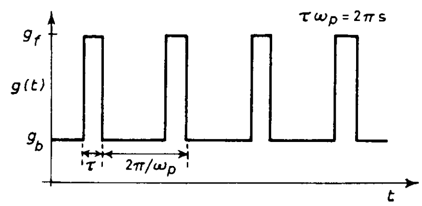

Fig. 4. Square-wave conductance variation.

Thus, using the notation of Section 2,

$$g_0 = g_b+(g_f-g_b)s \qquad \ldots\ldots(57)$$

$$g_1 = \frac{1}{\pi}\sin(\pi s)\,.\,(g_f-g_b) \qquad \ldots\ldots(58)$$

To establish the maximum value of g_1/g_0, the ratio is differentiated as a function of s and the result equated to zero. This leads to the result that g_1/g_0 is a maximum when

$$\tan \pi s = \frac{\pi[1+(n^2-1)s]}{n^2-1} \qquad \ldots\ldots(59)$$

This equation is difficult to solve analytically, but at least it is clear that for large n

$$\tan \pi s \simeq \pi s \qquad \ldots\ldots(60)$$

and therefore $s \to 0$, and the square wave consists of short pulses of high conductance, as shown in Fig. 4. Plots of g_1/g_0 for two values of n are given in Fig. 5.

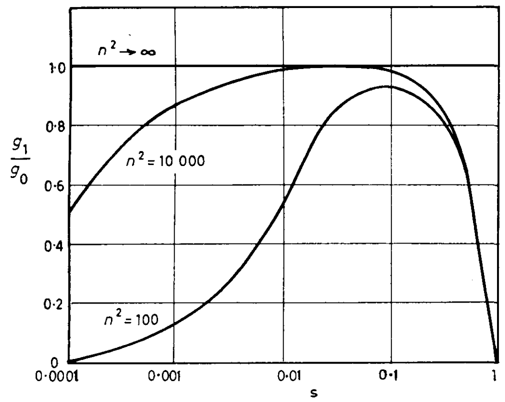

Fig. 5. Effect of variation of mark/space ratio of $g(t)$ upon g_1/g_0.

It can be seen that with a diode having a large ratio of forward to reverse conductance it is possible to have g_1/g_0 approach very closely to unity, and that the value of mark/space ratio necessary need not be controlled very accurately.

The corresponding result for the optimization of r_1/r_0 may readily be deduced. The desired square-wave variation of resistance consists of short pulses of high resistance.

5. Tentative Physical Explanations of Converter Operation

The up-converters described in previous sections have two remarkable properties: (i) the possibility of unidirectional operation; (ii) the increase in gain from that of the corresponding capacitive up-converter for the addition of a non-linear resistance which although varied, always remains positive.

A reasonable explanation for the unidirectional operation may be found in the field of polyphase modulators. For the circuit of Fig. 6 can be said to bear many resemblances to the circuit of section 2, if the non-linear capacitor plus the $\pi/2$ phase shifter in the upper branch be regarded as equivalent to a non-linear resistor (except on an energy-storage basis).

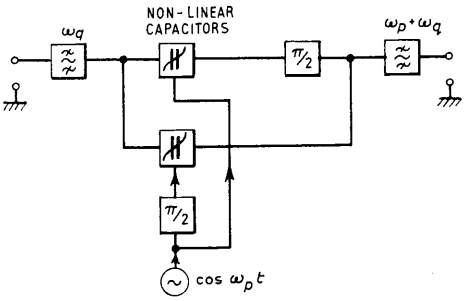

Fig. 6. Block schematic of polyphase upconverter (non-linear capacitor).

If the circuit blocks in Fig. 6 are regarded as being separated by buffer amplifiers which may be connected in whichever is the appropriate direction, a simple analysis may be undertaken.

For a voltage of $\cos \omega_q t$ applied at the left-hand terminals the output will be proportional to

$$\sin \overline{\omega_p+\omega_q t} + \sin \overline{\omega_p+\omega_q t} \qquad \ldots\ldots(61)$$

On the other hand, a voltage of $\cos \overline{\omega_p+\omega_q t}$ applied to the output terminals will give an output at the remote end of the device proportional to

$$\sin \omega_q t + \sin(-\omega_q)t = 0 \qquad \ldots\ldots(62)$$

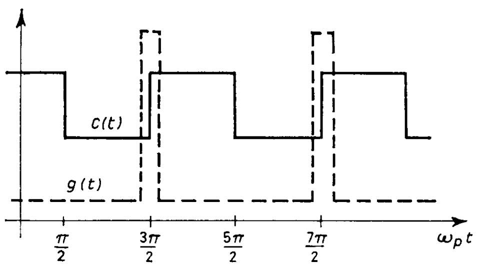

Fig. 7. Idealized capacitance and conductance variations.

so that unidirectionality has been achieved. It is suggested that the up-converters discussed earlier in this paper behave in this sense as approximations to such a polyphase circuit.

With regard to the increased circuit gain of the up-converters over more conventional capacitive types, Fig. 7 proves instructive. In this diagram, which is for the up-converter of section 2, it has been assumed that the capacitor has an equal mark/space square-wave variation of capacitance, and that the conductance variation has been optimized as discussed in the last section. The phasing has been chosen so that the up-converter gain is a maximum. If the resistance is considered to be idealized as a lossless switch, it can be seen that for most of a cycle, the up-converter is identical to a normal capacitive up-converter. Energy will be pumped into the circuit from the pump source when the capacitor decreases in value, since $q = Cv$, q cannot change instantaneously, and energy is proportional to Cv^2. There is no voltage across the capacitor and therefore no energy exchange when the capacitance increases, the switch having closed just before the capacitance rises. Hence energy is continually pumped into the circuit, in a more efficient way than for other conventional parametric amplifiers (except the degenerate amplifier), and thus the increased gain of the R-C circuit over conventional up-converters seems reasonable.

A similar argument can be put forward for the series combination of non-linear capacitance and resistance.

It is also possible to use an equivalent circuit representation of the circuit to obtain an understanding of its mode of operation. This method, although not so fundamental as the above, is capable of wider use.[12]

6. Experimental Work

A two-phase generator has been constructed, capable of supplying two pump voltages at the same frequency (150–500 kc/s) but with a relative phase shift of 0–180 deg between the channels. Figure 8 gives the simplified schematic of a parametric converter with parallel R-C elements which has been built. The R and C elements were constructed as balanced bridges in order to minimize the interaction of the two pump sources used to drive them. The non-linear resistors were G.E.C. silicon diodes, type SX 781SG, and the non-linear capacitances were I.R.C. silicon Zener diodes, type IZ 10. The latter had a zero-bias capacitance of about 1870 pF.

For the experimental work that followed, the pump frequency (ω_p) was chosen to be 300 kc/s, and the signal frequency (ω_q) to be between 15–50 kc/s. The circuit was first used without the non-linear resistor bridge and the power gain as a normal up-converter is recorded on Fig. 9. It will be seen that for high ratios of upper sideband to signal frequency the gain was higher than predicted. This was due to the difficulty found in rejecting adequately the corresponding lower sideband.

When the non-linear resistor bridge was added to the circuit and the two-phase generator was adjusted so that the R and C were pumped in quadrature, it

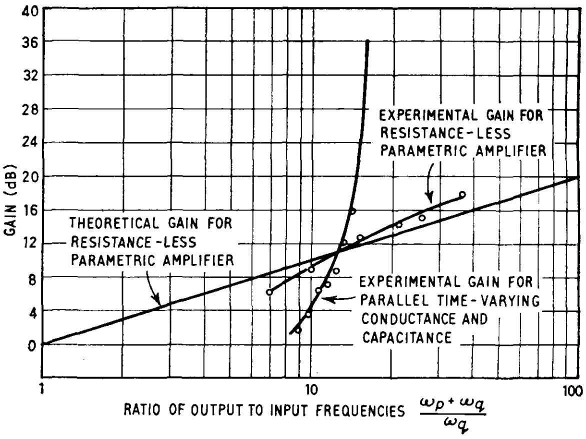

Fig. 9. Gain of parallel R-C converter.

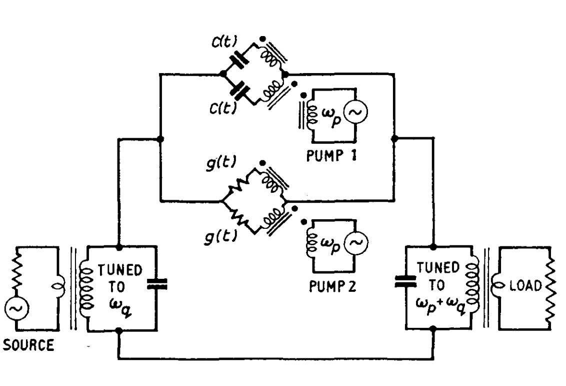

Fig. 8. Experimental parametric converter circuit.

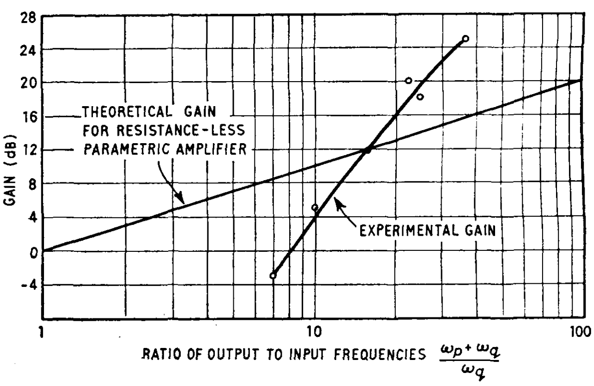

Fig. 10. Gain of series R-C converter.

was found possible to make the converter gain so high that the circuit oscillated for sideband to signal ratios between 8 and 15. It was not considered practicable to work at ratios greater than 15 because of the problem of rejecting the lower sideband. In order to compare the gain of the converter for different sideband frequencies, its performance was degraded by adding 3 kΩ resistors in series with each resistance diode. Under these conditions the near-vertical curve shown in Fig. 9 was obtained, and it can be seen that the gain varies with frequency in a manner that can be predicted from (29).† It was found exceptionally difficult to measure all the necessary circuit parameters, so that it is impossible to claim more than qualitative agreement.

A second experiment was performed in which the R and C of the circuit of Fig. 8 were connected in series rather than in parallel as in the work already described. This circuit is not readily analysed, being considerably more difficult to manipulate than the circuit described in Section 3. However, it was found that once again it was possible to obtain power gains greater than $(\omega_p+\omega_q)/\omega_q$ with the circuit, the results being plotted in Fig. 10.

7. Conclusions

The expression for the power gain of the up-converter using a parallel combination of non-linear capacitance and conductance has been found, for the condition of conjugate matching at the output termination only. It has been shown that when the input admittance has a negative real part the gain tends to infinity, and that this only happens when the conductance is pumped at a phase angle $\pi/2$ rad *behind* the capacitance. Infinite power gain is possible in principle for any value of pump frequency greater than zero. Under these conditions the reverse—i.e. down-conversion—gain also tends to infinity.

Previous results have also been confirmed, showing that the up-converter may be made unidirectional whilst still having a forward gain greater than $(\omega_p+\omega_q)/\omega_q$ for large ratios of output to input frequency. Since this gain is also dependent on the ratio of g_1/g_0, it has been shown how the latter ratio may be optimized by changing the mark/space ratio of the conductance variation so that for most of the pump cycle the diode has low conductance.

A corresponding analysis of the up-converter using a series combination of the same elements has shown similar results. In this case the resistance was pumped at a phase angle $\pi/2$ rad *ahead* of the elastance for optimum performance.

It has been shown that for either type of up-converter, if the resistance is pumped in antiphase to the preferred angle, only a degradation of power gain is recorded when the circuit is compared with a corresponding conventional up-converter without the non-linear resistance.

Attempts have been made to explain on physical grounds the unidirectional properties of the converters, by comparing them with a polyphase parametric converter. An argument for the large gains that are possible using the R-C converters is also given.

Experimental results are given which establish that both parallel and series R-C converters can be constructed which have significantly greater gain than corresponding capacitive converters. The results show that it is easier to obtain high gain in practical R-C converters if the pump frequency is much larger than the signal frequency.

It is shown in the Appendix that it is possible to use the type of converter discussed in this paper as a negative-resistance amplifier with a pump frequency below the signal frequency.

† This follows because the left-hand side of the equation, $g_0(G_0+g_0)$, is easy to make as large as desired. The right-hand side of the equation is much more difficult to increase relative to the left-hand side, since for a passive resistance $|g_1| \leqslant g_0$, and an increase in C_1 will in practice eventually result in an increase in G_0. (If the capacitive diode is considered to be an ideal non-linear capacitor in parallel with a linear resistance, the latter will appear as part of G_0 and G_{+1}.) Hence a progressive reduction of ω_p, lessening the ratio of output to input frequencies in Fig. 9, will eventually make it impossible to achieve arbitrarily-high gain in a given circuit, and the gain will thereafter decrease in the manner shown.

8. Acknowledgments

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10. Appendix

From equation (22)

Y_{+1} has a negative real part if

$$(\omega_q C_1 - g_1)(\overline{\omega_p + \omega_q} C_1 + g_1) + g_0(g_0 + G_0) < 0 \qquad \ldots\ldots(63)$$

if the conductance is pumped $\pi/2$ rad *behind* the capacitance.

Hence

$$g_1 > \omega_q C_1 \qquad \ldots\ldots(64)$$

and the device is not unidirectional. However, it may be used as a negative-resistance amplifier, considering the signal circuit to be the *output* loop. In this case the pump frequency will be below the signal frequency.

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The discussion on this paper starts on page 439

STANDARD FREQUENCY TRANSMISSIONS

(Communication from the National Physical Laboratory)

Deviations, in parts in 10^{10}, from nominal frequency for **May 1964**

May 1964	GBR 16kc/s 24-hour mean centred on 0300 U.T.	MSF 60 kc/s 1430–1530 U.T.	Droitwich 200 kc/s 1000–1100 U.T.	May 1964	GBR 16 kc/s 24-hour mean centred on 0300 U.T.	MSF 60 kc/s 1430–1530 U.T.	Droitwich 200 kc/s 1000–1100 U.T.
1	− 150·4	− 151·2	+ 1	17	− 150·3	− 151·3	+ 5
2	− 150·8	− 151·0	+ 2	18	− 150·6	− 151·3	+ 4
3	− 151·5	− 151·7	+ 2	19	− 151·3	− 149·5	+ 4
4	− 151·5	—	0	20	− 149·6	− 149·8	+ 4
5	− 151·4	− 150·6	+ 1	21	− 150·5	− 150·0	+ 5
6	− 150·7	—	0	22	− 150·4	− 151·9	+ 6
7	—	− 150·8	+ 1	23	− 150·7	− 151·5	+ 4
8	− 150·2	− 150·4	− 1	24	− 151·4	− 152·6	+ 2
9	− 150·2	− 150·7	—	25	− 151·1	− 150·4	+ 2
10	− 150·0	− 149·1	+ 2	26	—	− 152·6	+ 3
11	− 150·2	− 150·4	+ 2	27	− 150·1	—	+ 4
12	− 150·7	− 150·0	+ 1	28	− 150·0	− 150·3	+ 5
13	− 151·1	− 149·5	+ 1	29	− 150·6	− 151·5	+ 3
14	− 150·9	− 152·6	+ 2	30	− 151·6	− 151·9	+ 5
15	− 150·8	− 151·0	+ 2	31	− 152·0	—	+ 3
16	− 150·0	− 150·6	+ 4				

Nominal frequency corresponds to a value of 9 192 631 770 c/s for the caesium F,m (4,0)–F,m (3,0) transition at zero field.