Chapter 6. Maxwell's Equations, Macroscopic Electromagnetism, Conservation Laws

6.1 Maxwell Displacement Current and Maxwell Equations

Differential equations for calculating fields from currents and charges – what we have so far

	Vacuum	Medium	
Inhomogeneous	$\nabla \cdot \mathbf{E} = \rho/\varepsilon_o$	$\nabla \cdot \mathbf{D} = \rho$	(6.1a)
	$\nabla \times \mathbf{B} = \mu_o \mathbf{J}$	$\nabla \times \mathbf{H} = \mathbf{J}$	(6.1b)

Homogeneous

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \qquad (6.1c)$$

$$\nabla \cdot \mathbf{B} = 0 \qquad (6.1d)$$

These were derived from electrostatics and magnetostatics, plus the $\partial \boldsymbol{B}/\partial t$ term that was added to incorporate Faraday's experiments.

James Clerk Maxwell was the first to actually write (6.1), I think. He asked himself whether anything else needed to be fixed to make the equations consistent as a complete theory of time-dependent electromagnetism.

One clear thing to check is to take the divergence of a curl. Apply to (6.1c)

$$\nabla \cdot (\nabla \times \mathbf{E}) + \frac{\partial (\nabla \cdot \mathbf{B})}{\partial t} = 0$$

(6.1d) is almost not an independent equation. Equation (6.1c) implies that

$$\frac{\partial (\nabla \cdot \mathrm{B})}{\partial t} = 0$$

so (6.1d) just states that the constant value of $\nabla \cdot \mathbf{B}$ is zero.

Taking the divergence of (6.1b) gives

$$\nabla \cdot \mathbf{J} = \mathbf{0}$$

Conservation of charge, however, implies

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{6.3}$$

so (6.1b) is inconsistent with conservation of charge.

Can we add a correction term to (6.1b)?

$$\nabla\times\mathbf{B} = \mu_o\mathbf{J} + \mathbf{f} \qquad \text{or} \qquad \nabla\times\mathbf{H} = \mathbf{J} + \mathbf{g}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$0 = \mu_o \nabla\cdot\mathbf{J} + \nabla\cdot\mathbf{f} \qquad\qquad 0 = \nabla\cdot\mathbf{J} + \nabla\cdot\mathbf{g}$$

using (6.3)

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$0 = -\mu_o \frac{\partial \rho}{\partial t} + \nabla\cdot\mathbf{f} \qquad\qquad 0 = -\frac{\partial \rho}{\partial t} + \nabla\cdot\mathbf{g}$$

What has a divergence equal to $\frac{\partial \rho}{\partial t}$? From (6.1a)

$$\nabla\cdot\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\varepsilon_o}\frac{\partial \rho}{\partial t} \qquad\qquad \nabla\cdot\frac{\partial \mathbf{D}}{\partial t} = \frac{\partial \rho}{\partial t}$$

$$0 = -\mu_o\varepsilon_o\nabla\cdot\frac{\partial \mathbf{E}}{\partial t} + \nabla\cdot\mathbf{f} \qquad\qquad 0 = -\nabla\cdot\frac{\partial \mathbf{D}}{\partial t} + \nabla\cdot\mathbf{g}$$

Let's try $\quad \mathbf{f} = \mu_o\varepsilon_o\frac{\partial \mathbf{E}}{\partial t} \qquad\qquad \mathbf{g} = \frac{\partial \mathbf{D}}{\partial t}$

Then the revised versions of the Ampere's-law Maxwell equation become

$$\nabla\times\mathbf{B} = \mu_o\mathbf{J} + \mu_o\varepsilon_o\frac{\partial \mathbf{E}}{\partial t} \tag{X6.1}$$

$$\nabla\times\mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \tag{6.6b}$$

Are these consistent? Recall that $\quad \mathbf{D} = \varepsilon_o\mathbf{E} + \mathbf{P} \qquad (4.34)$

$\uparrow$

electric polarization, dipole moment/volume

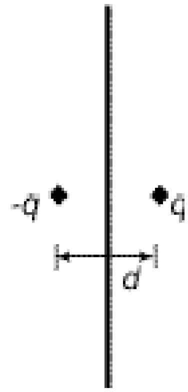
Equation (6.6b) becomes

$$\nabla\times\mathbf{H} = \mathbf{J} + \varepsilon_o\frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} \tag{X6.2}$$

Is there a physical interpretation of the $\partial\mathbf{P}/\partial t$ term? Consider a simple picture of the medium, with each dipole in the polarizable medium pictured as charges $\pm q$ separated by a vector $\mathbf{d}$, which changes in time.

Number of charges/area crossing plane

$\mathbf{P} = Nq\mathbf{d}$



$$Nq \frac{\partial \mathbf{d}}{\partial t} dt = \frac{\partial \mathbf{P}}{\partial t} dt$$

So $\partial\mathbf{P}/\partial t$ is a current density associated with time changing polarization. It's called the *polarization current*

$$\mathbf{J}_P = \frac{\partial \mathbf{P}}{\partial t} \tag{X6.3}$$

Note that $\mathbf{J}_P$ satisfies

$$\nabla\cdot\mathbf{J}_P + \frac{\partial \rho_P}{\partial t} = 0$$

where $\rho_P = -\nabla\cdot\mathbf{P}$ is the polarization charge density defined earlier. The extra $\partial\mathbf{P}/\partial t$ term (X6.2) therefore isn't mysterious—it's physically unavoidable.

We have identified 3 kinds of currents.

- Conduction - $\mathbf{J}$
- Magnetization - $\nabla \times \mathbf{M}$
- Polarization - $\partial\mathbf{P}/\partial t$

So what is $\varepsilon_o\, \partial\mathbf{E}/\partial t$ physically? Maxwell *et al.* called this the "vacuum polarization current." He visualized the vacuum as a ponderable medium that polarized in response to a changing $\mathbf{E}$.

How big an effective current density? Assume $E \sim 100$ V/m, $t \sim 1$s

$$\varepsilon_o \frac{\partial E}{\partial t} = 8.85 \times 10^{-12} \times \frac{100}{1} = 8.85 \times 10^{-10} \frac{\text{amp}}{\text{m}^2}$$

That's why Faraday and his friends didn't discover the vacuum-displacement-current term experimentally. In experiments with wires and batteries and magnets, this vacuum displacement current is really, really small. Faraday wouldn't have detected it.

Maxwell's Equations (Now in all their glory)

<u>Free Space</u>	<u>Medium</u>	
$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_o}$	$\nabla \cdot \mathbf{D} = \rho$	(6.6a)
$\nabla \times \mathbf{B} = \mu_o \mathbf{J} + \mu_o \varepsilon_o \frac{\partial \mathbf{E}}{\partial t}$	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	(6.6b)

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \tag{6.6c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{6.6d}$$

Once Maxwell had his equations, he decided to take them out for a spin. Do they have any interesting solutions in a pure vacuum ($\rho = \mathbf{J} = \mathbf{M} = \mathbf{P} = 0$)?

$$\left.\begin{aligned} &\nabla \cdot \mathbf{E} = 0 \\ &\nabla \times \mathbf{B} = \mu_o \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} \\ &\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0 \\ &\nabla \cdot \mathbf{B} = 0 \end{aligned}\right\} \tag{X6.4}$$

Take curl of (X6.4b) and use (X6.6c):

$$\nabla \times (\nabla \times \mathbf{B}) = -\mu_o \varepsilon_o \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

$$\nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\mu_o \varepsilon_o \frac{\partial^2 \mathbf{B}}{\partial t^2}$$

$$\nabla^2 \mathbf{B} - \mu_o \varepsilon_o \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0 \tag{X6.5}$$

This is the wave equation. Try a wave solution

$$\mathbf{B} = \mathbf{B}_o \cos(\mathbf{k} \cdot \mathbf{x} - \omega t)$$

$$-k^2 \mathbf{B}_o + \omega^2 \mu_o \varepsilon_o \mathbf{B}_o = 0$$

This wave solution implies the *dispersion relation*

$$\frac{\omega^2}{k^2} = \frac{1}{\mu_o \varepsilon_o} = (\text{wave velocity})^2 \tag{X6.6}$$

$$\frac{1}{\sqrt{\mu_o \varepsilon_o}} = \frac{1}{\sqrt{4\pi \times 10^{-7} \times 8.85 \times 10^{-12}}} = 2.999 \times 10^8 \frac{\text{m}}{\text{s}} = c \tag{X6.7}$$

Maxwell didn't use the same units. In his units these were measurement-based constants representing the strength of the electric and magnetic forces. However, the Maxwell's result was basically the same.

So, combining the equations of electrostatics, plus Faraday's law + adding the vacuum polarization term for consistency with conservation of charge, Maxwell found a set of partial differential equations that, when applied to a vacuum, reduced to the wave equation. The wave velocity, dependent on the strength constants for electrostatics and magnetostatics, turns out to be $3{\times}10^8$ m/s, which was (and is) the measured speed of light.

The clear implication was that this theory of electromagnetism also describes light. When Maxwell realized this, it must have been one of the great moments of theoretical physics.

USE OF VECTOR AND SCALAR POTENTIALS IN MAXWELL'S EQUATIONS

The reason for using vector and scalar potentials is that their use guarantees that the homogeneous MEs

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{6.6c}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{6.6d}$$

are satisfied. The assumption that

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{6.7}$$

introduced for magnetostatics, guarantees satisfaction of (6.6d). The assumption

$$\mathbf{E} = -\nabla\Phi$$

from electrostatics needs modification for electrodynamics, because it implies $\nabla \times \mathbf{E} = 0$. Try

$$\mathbf{E} = -\nabla\Phi + \mathbf{f}$$

in (6.6c)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{f}$$

Since $\mathbf{B} = \nabla \times \mathbf{A}$, it makes sense to set $\mathbf{f} = -\frac{\partial \mathbf{A}}{\partial t}$

$$\mathbf{E} = -\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t} \tag{6.9}$$

That leaves us solving for 4 functions (Φ, A_x, A_y, A_z) instead of 6 (E_x, E_y, E_z, B_x, B_y, B_z).

What do the inhomogeneous Maxwell equations imply for ϕ and $\mathbf{A}$?
These potentials are useful primarily for the free-space form of Maxwell's equations. Substituting (6.7) and (6.9) in the free-space versions of (6.6a) and (6.6b) gives

$$\nabla\cdot\mathbf{E} = \nabla\cdot\left(-\nabla\Phi - \frac{\partial \mathbf{A}}{\partial t}\right) = \rho/\varepsilon_o$$

$$\nabla^2\Phi + \frac{\partial(\nabla\cdot\mathbf{A})}{\partial t} = -\rho/\varepsilon_o \tag{6.10}$$

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_o\mathbf{J} + \mu_o\varepsilon_o\left[-\frac{\partial\, \nabla\Phi}{\partial t} - \frac{\partial^2\mathbf{A}}{\partial t^2}\right] = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} - \nabla\left[\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\Phi}{\partial t}\right] = -\mu_o\mathbf{J} \tag{6.11}$$

where I used (X6.7).

This is 4 equations to be solved for 4 unknowns. But (6.10) and (6.11) are UGLY. They're linear, but they're coupled, and they aren't pretty. This can be fixed with the Lorentz gauge condition (next time).