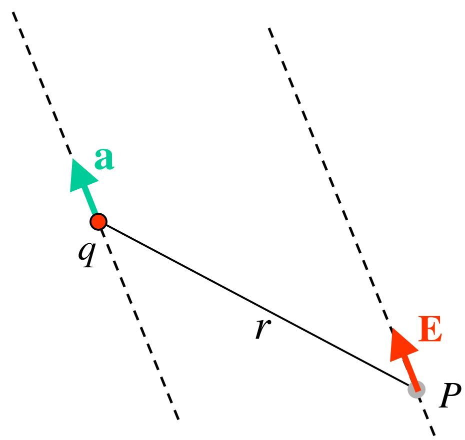
DC Induction into a Closed Loop from Accelerating Charge

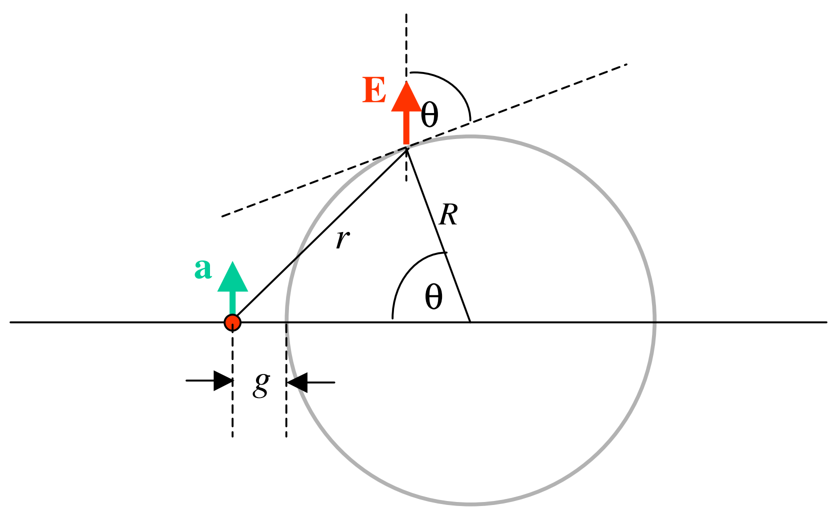
An accelerating charge q radiates an electric field **E** at any point P given by

$$\mathbf{E} = \frac{\mu_0 q \mathbf{a}}{4\pi r} \qquad (1)$$

where **a** is the acceleration and r the distance from the charge to the point P. Here bold type represents vectors. This is the classical radiation law where the field is proportional to inverse distance (power proportional to the square of the inverse distance). It should really contain a retardation feature to take account of the finite propagation delay, but since we will be working in the very near field we can neglect retardation effects. It tells us that the radiation is isotropic, it radiates equally in all directions, and the **E** field is always parallel to the acceleration direction.



The isotropic nature makes the math easy in our case where we wish to establish the voltage induced into a circular loop.



With the acceleration in the plane of the loop at a distance g from its outer edge and parallel to the tangent there we see that by the cosine rule at any point on the loop the distance r is given by

$$r = \sqrt{R^2 + (R+g)^2 - 2R(R+g)\cos(\theta)} \qquad (2)$$

where R is the radius of the loop.

The component of **E** tangential to the loop is $E\cos(\theta)$ and that induces a voltage increment of $E\cos(\theta)R.\mathrm{d}\theta$ into an elemental length $R.\mathrm{d}\theta$ of the circumference. Hence the total voltage induced into the loop is given by

$$V = \frac{\mu_0 qaR}{4\pi}\left[\int_0^{2\pi} \frac{\cos(\theta)}{\sqrt{R^2 + (R+g)^2 - 2R(R+g)\cos(\theta)}} \cdot \mathrm{d}\theta\right] \qquad (3)$$

The solution to this integral is quite complex and involves elliptic integals of the first and second kind. A simple method can be performed on a spreadsheet taking 1° increments, solving for each increment then summing the 360 results,

$$V = \frac{\mu_0 qaR}{720}\left[\sum_{n=0}^{n=359} \frac{\cos(n)}{\sqrt{R^2 + (R+g)^2 - 2R(R+g)\cos(n)}}\right] \qquad (4)$$

where here the angle n is expressed in degrees.

For our case where the acceleration occurs as the electrons move from a stationary brush onto a rotating slip-ring, we need to know the amount of charge that is accelerating at any instant in time. Because of the trivial drift velocity in the brush compared to the surface velocity v of the slip-ring we can assume that electrons go from near zero velocity up to that surface velocity over the width w of the brush (we will find later that w disappears from the calculations). Hence the acceleration a is given by $v^2/2w$. With a current i flowing, remembering that $i=\mathrm{d}q/\mathrm{dt}$ and taking the mean velocity across the acceleration region as $v/2$ we obtain $q = 2iw/v$. Hence the product qa is given by

$$qa = iv \qquad (5)$$

Putting this into (4) gives us the final formula to use in the spread sheet.

$$V = \frac{\mu_0 ivR}{720}\left[\sum_{n=0}^{n=359} \frac{\cos(n)}{\sqrt{R^2 + (R+g)^2 - 2R(R+g)\cos(n)}}\right] \qquad (6)$$