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ON THE OSCILLATIONS OF A CIRCUIT HAVING A PERIODICALLY VARYING CAPACITANCE*

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Summary—*A previous theoretical study of a dissipationless oscillatory circuit having a periodically varying capacity predicted the existence of several interesting types of oscillations. An experimental investigation of such a circuit was therefore made. The results of this investigation are presented in this paper. The dissipation was reduced to a minimum by means of an associated vacuum tube in a regenerative connection. An auxiliary condenser and rotating commutator produced the periodic capacity variation. The nature of the oscillations was studied by means of oscillograph and meter measurements of the currents in the several parts of the circuit and by the use of headphones. A quite good agreement with theory was found, consistent with the deviations of the experimental conditions from the theoretically postulated ones. The oscillations were generally of a complicated nonsinusoidal character, but assumed a substantially sinusoidal form for certain adjustments of the circuit. The frequency of these sinusoidal oscillations was twice that of the capacity variation. Periodic oscillations were found to occur when, approximately,* $\alpha = 2\omega_0/n$, $n = 1, 2, 3, \cdots$, *where* ω_0 *is the natural angular velocity of the circuit for the mean capacity value and* α *is the angular velocity of the capacity variation. Failure of all oscillations for certain circuit adjustments was related to the inherent loss of energy in the switching method employed for varying the capacity. The theory had predicted certain unstable conditions for which the amplitude should increase without limit. In the experimental circuit these unstable adjustments gave a large amplitude sinusoidal oscillation.*

Introduction

In a previous paper[1] a mathematical study was made of the oscillations of a dissipationless oscillatory circuit having a periodically varying capacity. The existence of several very interesting phenomena was predicted from the analysis, and it was pointed out that their occurrence in any physically realizable circuit could not exactly follow the theory because of the absence of any form of dissipation in the idealized elements on which this study was based. Inasmuch as no experimental investigation of these effects was known to exist, one was undertaken[2] with the object of further determining the performance of

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[1] W. L. Barrow, "Frequency modulation and the effects of a periodic capacity variation in a nondissipative oscillatory circuit," Proc. I.R.E., vol. 21, pp. 1182–1202; August, (1933).

[2] Mr. S. Sorkin and Mr. L. Jacobson, seniors in the electrical engineering department, communications option, Massachusetts Institute of Technology, were of material assistance in carrying out the experimental work.

the circuit in question. The present paper presents the most important results of this experimental study in which the theoretical analysis is corroborated.

Theoretical Discussion

For the details of the theory of the dissipationless resonant circuit with periodically varying capacity, reference may be made to the earlier paper,[1] as well as other citations given there. It is nevertheless of interest to review briefly the results of the theoretical analysis before proceeding to the experimental work.

Denoting the charge by Q, the mean frequency[3] of the circuit about which the variation takes place by ω_0, the percentage change in frequency by h, the frequency of variation of the capacity by α, and the time by t, the differential equation takes the form

$$\frac{d^2Q}{dt^2} + (\omega_0^2 + h\cdot\omega_0^2 \cos \alpha t)Q = 0 \tag{1}$$

provided a capacity variation is assumed in which

$$1/C(t) = \frac{1}{C_0} + \frac{1}{C_0}\cdot h \cos \alpha t,$$

which may be considered as a first approximation to the particular variation used in the experiment. It can be shown[3] that the solution of (1) is the real part of

$$Q = C_1 e^{-j\mu\alpha t}\cdot \sum_{n=-\infty}^{+\infty} b_n e^{jn\alpha t} + C_2 e^{+j\mu\alpha t}\cdot \sum_{n=-\infty}^{+\infty} b_n e^{jn\alpha t} \tag{2}$$

where C_1, C_2 are arbitrary constants determined by the initial conditions, and $j=\sqrt{-1}$. The summation is a Fourier series written in complex form, the coefficients b_n of which are to be determined from the recurrence relation:

$$\left[\left(\frac{\omega_0}{\alpha}\right)^2 - (\mu + n)^2\right] b_n + \frac{1}{2} h \left(\frac{\omega_0}{\alpha^2}\right)^2 [b_{n-1} + b_{n+1}] = 0. \tag{3}$$

The factor μ in the exponent of (2) must also be determined from (3) and is called the "characteristic exponent." In general, μ is a complex number $u+jv$, but for certain values of the parameters may be simply a real number. From a consideration of (2), this fact is seen to be of prime importance, for, when μ is real, the solution takes the form

[3] For brevity in text and formulas the word *frequency* will be used throughout this paper to mean angular velocity $=\omega=2\pi f$.

$$Q = C_1 \sum_{-\infty}^{+\infty} b_n \cos (n - \mu)\alpha t + C_2 \sum_{-\infty}^{+\infty} b_n \cos (n + \mu)\alpha t, \tag{4}$$

which is a steady state solution composed of several simultaneous (theoretically, an infinite number) sinusoidal oscillations of different frequencies and amplitudes. On the other hand, when μ has an imaginary part different from zero, say $\mu = u + jv$, the solution becomes

$$Q = C_1 e^{+v\alpha t} \cdot \sum_{-\infty}^{+\infty} b_n \cos (n - u)\alpha t + C_2 e^{-v\alpha t} \cdot \sum_{-\infty}^{+\infty} b_n \cos (n + u)\alpha t. \tag{5}$$

The presence of the exponential factors in (5) causes the second term to become negligibly small compared to the first after a short interval

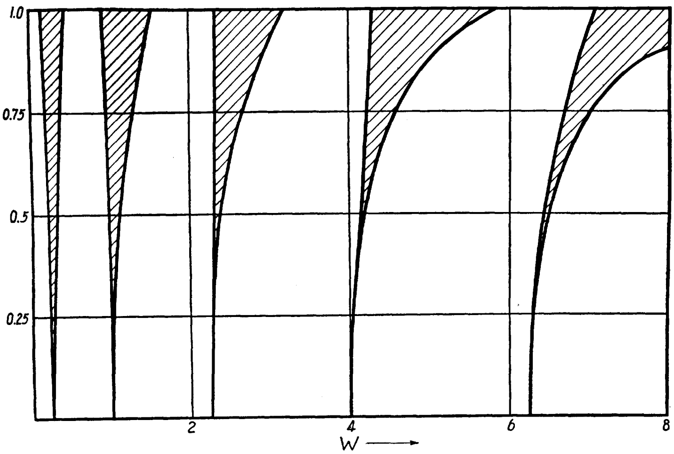

Fig. 1—Showing ranges of ω_0, α, and h for which oscillations of stable (unshaded) and unstable (shaded) type are theoretically expected to occur. [Fig. 6, Proc. I.R.E., vol. 21, no. 8, p. 1195; August, (1933)] $W = (\omega_0/\alpha)^2$.

of time, leaving only the first term, which itself *increases in magnitude without limit.* This type of solution is called "unstable" and obviously cannot occur in precisely this form in any physically realizable circuit. The factors preventing such unstable oscillations depend, naturally, on the particular apparatus used in constructing the circuit; here, for example, the nonlinear characteristics of the associated vacuum tube are quite sufficient to explain a limited amplitude of oscillation, and thus a departure from the mathematical form of instability is to be expected. It is one of the purposes of this study to determine the nature of the oscillations in an actual circuit for an unstable adjustment. For stable adjustments, the solution (5) is closely related to that in (5); in fact, the elimination of the second term in (5), due to the rapid damping of the negative exponential, suggests a simpler type of oscillation. A further simplification of the oscillation of type given by (5) follows from the fact that μ is a constant over the whole range in which $v \neq 0$. A graphical representation of the relation between μ and the circuit parameters is given in Fig. 1, and is of considerable importance for the following experimental work, as it defines the ranges of values for ω_0/α and h for which oscillations of the several kinds are expected to take place. It is particularly to be noted that on the boundaries between the two kinds of oscillations the mathematical solution indicates a periodic oscillation of period α or $\alpha/2$.

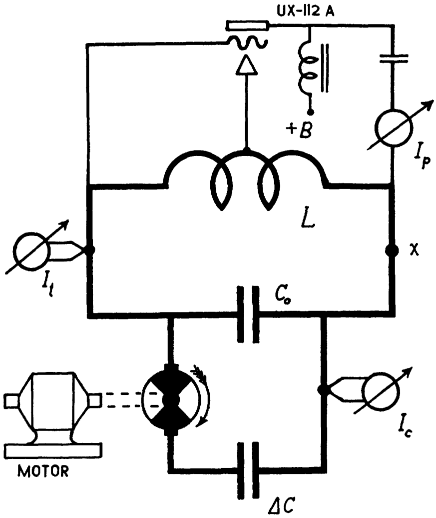

Fig. 2—The experimental circuit. $L = 1.3$ henrys, C_0 and ΔC had values as given in text.

The Experimental Circuit and Measurements

In performing the experimental investigation two main deviations from the ideal case treated in the theory were made, namely: (1) a regenerative vacuum tube circuit was used to simulate the dissipationless L, $C(t)$ circuit; and, (2) a square-wave type of capacity variation was used instead of a sinusoidal one. A regenerative vacuum tube oscillator presents an easily realizable circuit[4] and probably approximates a dissipationless one as well as this can be done today. Deviation (1) may be held responsible for such observed differences between theory and experiment as, for example, the failure to obtain any oscillation for $h > 0.3$. As previously mentioned, the nonlinear characteristics of the

[4] A dynatron oscillator is another very interesting connection which might be used to study further the properties of the $L \cdot C(t)$ circuit.

vacuum tube, presence of grid current, etc., tend to give a circuit whose average resistance over a complete cycle is zero, rather than one having a zero resistance for every instant of time. Phenomena different from those predicted on the basis of the idealized circuit are therefore to be expected. The second deviation was one imposed by conditions of mechanical feasibility, as a rotating plate condenser of the desired size would be inconveniently large, costly, and very difficult to vary with sufficient rapidity. As previously pointed out, the sinusoidal variation may be considered as a good first approximation; besides, it has been shown that very little difference is to be expected between the oscillations for a sinusoidal type of variation and those for a square one.[1,5]

The circuit used, together with the values of the parameters, is reproduced in Fig. 2. A commutator driven by a small direct-current motor with rheostats for adjusting the speed was used to produce the periodic capacity variation by alternately connecting and disconnecting the auxiliary condenser ΔC. The commutator was also equipped with an auxiliary contact mechanism for registering on the oscillogram the periods of time during which ΔC was connected. Two commutators having two and six segments, respectively, were used. Thermal ammeters indicated the current I_t in the oscillatory circuit and the current I_c between C_0 and ΔC. An oscillograph vibrator of one ohm resistance could be inserted at x. The frequency of the capacity variation was measured by measuring the commutator speed with a precision tachometer.

Results

Data were taken in several forms, namely: r-m-s values of the currents I_t, I_c, and I_p, oscillograms of wave form and of current envelope, and the comparative auditory sensation observed by listening to headphones connected to an inductive pick-up coil. The latter was useful for rapidly classifying the oscillation as "apparently sinusoidal," "slow beats," etc. It may be seen from (1) and from the accompanying discussion that h is defined in terms of the capacities C_0 and ΔC as

$$h = \frac{\Delta C/2}{C_0 + \Delta C/2} \cdot \tag{6}$$

The inductance was held constant in these experiments and ΔC and C_0 both varied to obtain different values of h in such a manner that ω_0 remained unchanged. In this way the circuit performance was studied

[5] See in particular van der Pol and Strutt, *Phil. Mag.*, vol. 5, p. 18; (1928). The results of van der Pol and Strutt give an exact treatment of a square-wave variation, but the information available for the sinusoidal case is considerably more complete, hence its use here.

for values of h up to 0.23 and $(\omega_0/\alpha)^2$ up to 9.0 for a constant $\omega_0=97.5$ c.p.s. Above $h=0.23$ no oscillations took place, due no doubt to the fact that the regenerative conditions for the vacuum tube were then too unfavorable to allow self-maintained oscillations to occur.

As h and α, and thus $(\omega_0/\alpha)^2$, were varied, the character of the oscillations also varied, but certain critical points and regions were found where the nature of the oscillations was outstanding. Specifically, upon varying α so as to approach one of these critical regions, rapid beats, i.e., waxing and waning, in the current were observed. These beats gradually became slower in period but larger in magnitude until the particular critical point was reached, whereupon the amplitude assumed a large value and the wave form became substantially sinu-

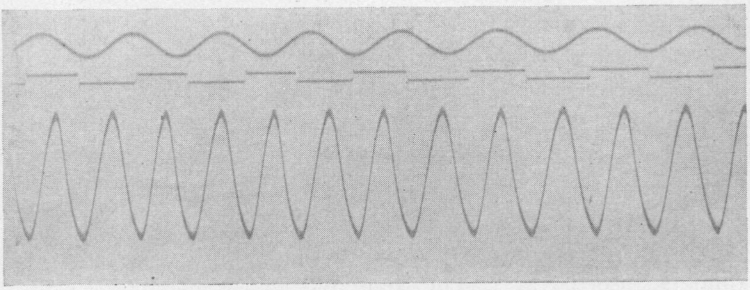

Fig. 3—Oscillogram of the current (the lower trace c) of substantially sinusoidal wave form for a critical (unstable) adjustment of the circuit parameters; $\alpha=2\omega_0$. The upper trace a is a 60-cycle per second timing wave. In the second trace b ΔC is connected when the horizontal line is above and disconnected when it is below.

soidal. A series of such points or regions were located at values of α of $2\omega_0$, ω_0, $0.66\omega_0$, $0.5\omega_0$, $0.25\omega_0$, $\cdots$, corresponding to values of (ω_0/α) of 0.5, 1.0, 1.5, 2.0, 2.5, $\cdots$, respectively. A glance at Fig. 1 shows that these are precisely the circuit adjustments for which either periodic or unstable oscillations were to be expected. It is thus concluded that instability of the ideal circuit is manifested in the actual circuit by an increase in amplitude and a wave form of substantially sinusoidal character. A typical oscillogram of the current in the oscillatory circuit for a critical adjustment is reproduced in Fig. 3 and shows plainly the character of the oscillation. When one considers the abrupt change in capacity occurring in the circuit once during each period, it seems quite remarkable that the oscillations should be of this simple sinusoidal form.

In passing, it is interesting to note that the unstable condition existing when $\alpha=2\omega_0$ was first noticed in connection with the vibrations of a string whose tension was given a periodic variation, and was discussed theoretically by Lord Rayleigh[6] in 1887. If the capacity of the electrical circuit was varied at this rate by a method not involving a considerable loss of energy (for instance, by a rotating condenser),

a
b
c

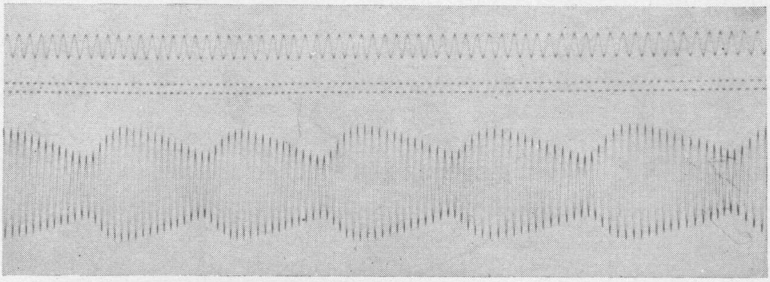

Fig. 4—Oscillogram of the beat type of oscillation occurring for a circuit adjustment differing only slightly from a critical (unstable) value.

self-maintained oscillations would be set up in the circuit without the help of a vacuum tube. Winter-Günter[7] has produced such oscillations by periodically varying the inductance instead of the capacity.

Fig. 4 shows a typical oscillogram of the current having the relatively slow beats just described above and Fig. 5 shows a more complex

a
b
c

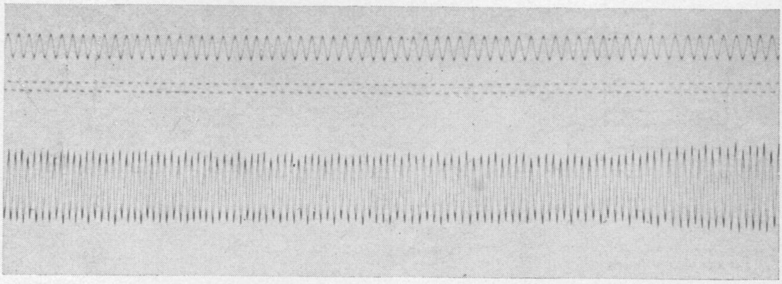

Fig. 5—Oscillogram of the complex wave form typical of an adjustment to the middle of a stable area of Fig. 1. Approximately $h=0.15(\omega_0/\alpha)^2=1.5$.

wave form occurring about midway between two critical points. The occurrence of the beat phenomena may be explained from the theory, for the solution on the boundary between a stable and unstable region is known to be purely periodic, as μ in (4) is either an integer or half integer. Further, it is practically sinusoidal for not too large values of h as the coefficients b_n then converge very rapidly. When the adjust-

[6] Scientific Papers, vol. III.
[7] *Jahr. der draht. Tel. u. Tel.*, vol. 34, p. 1, (1929), and vol. 37, p. 172, (1931).

ment is not quite that for a periodic solution, μ differs only slightly from an integer or half integer, and components of current exist which differ only slightly in frequency, producing the beats observed. For example, if Δ represents a number—small compared with unity—and $\mu = 1 + \Delta$, the frequencies as given by (5) are

$$[n - (1 + \Delta)]\alpha, \qquad [n + (1 + \Delta)]\alpha, \qquad n = \begin{cases} +1, 2, 3 \cdots \\ 0, -1, -2, \cdots \end{cases}$$

and beats having a lowest frequency $2 \cdot \Delta\alpha$ occur. These beats vanish for $\Delta = 0$ and become more and more rapid as Δ increases. The observed

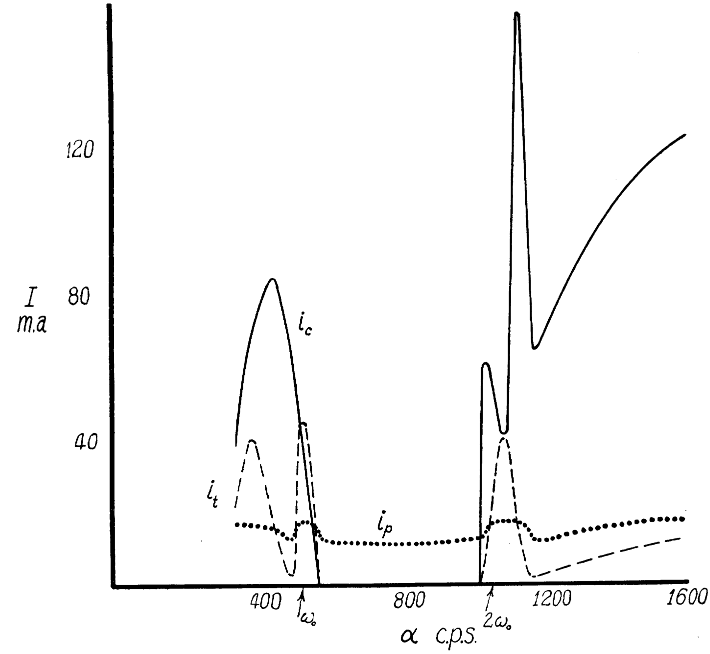

Fig. 6—Curves of currents in the oscillatory circuit i_t, between condensers i_c and in plate circuit of the vacuum tube i_p for $C_0 = 2$ microfarads and $\Delta C = 1$ microfarad.

increase in amplitude for the critical values may also be inferred from (5), which is pertinent to the case. The amplitude of the oscillation builds up as demanded by the positive exponential until the non-linearity of the vacuum tube characteristics causes a stable amplitude to be maintained; the wave form is then the same as that of the periodic oscillation just preceding instability.

The occurrence of periodic oscillations having a fundamental frequency different from α or $\alpha/2$ for other than these critical values was predicted from the theory and such oscillations were actually observed. The wave form showed the presence of many simultaneous harmonic oscillations in each case, as contrasted with the substantially sinusoidal wave of the critical point oscillations.

When h was given a relatively large value greater than 0.16 and α varied continuously from zero upward, it was found that for several ranges of values of α the oscillations ceased altogether, even though sinusoidal oscillations of considerable amplitude took place in between these ranges. The ranges of α in which oscillations cease have been located as the values of $(\omega_0/\alpha)^2$ corresponding to the stable areas of Fig. 1. Curves showing the behavior of the currents in the oscillatory circuit and plate circuit of the vacuum tube for a typical run are given in Fig.

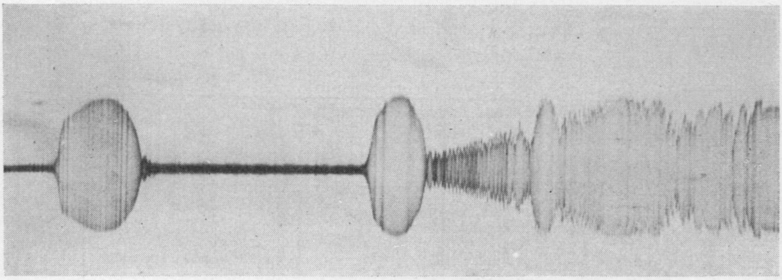

Fig. 7—Oscillograms showing the character of oscillations for $C_0 = 1.7$, $\Delta C = 1.0$ microfarad, and $L = 1.3$ henrys. The speed of the motor was continuously changed during the taking of the oscillogram such that $(\omega_0/\alpha)^2$ went from an infinite value at the right (i.e. zero r.p.m. of commutator) to about 0.5 at the left. Unevenness of paper accounts for the irregular blackening observable in the oscillogram.

6. The oscillogram of Fig. 7 makes this interesting phenomenon for relatively larger values of h even clearer. This oscillogram was made by registering the current in the tuned circuit (at a speed too slow to resolve the individual oscillations) as the frequency of capacity variation α was continuously varied from zero to more than $2\omega_0$. Two regions within which oscillations ceased, the large amplitude sinusoidal oscillations in the critical (unstable) regions and the adjacent regions of beats with the highly complex wave forms in between, may all be distinctly seen. It is thought that the explanation of this interesting phenomenon lies generally in the fact that in the region of no oscillations the condensers C_0 and ΔC have large voltages of opposite polarity at the instant of connection resulting in a loss of energy when these voltages are equalized (a familiar proposition of electrodynamics and circuit theory), greater than the vacuum tube can supply. This explanation is supported by the observed fact that the current I_c between

condensers is zero in the critical regions and of considerable magnitude in between these regions.

An oscillogram similar to that of Fig. 7 is reproduced in Fig. 8. Here, the magnitude of h was only 0.13, and consequently oscillations continued for all rates of capacity variation. The several types of oscillations are again clearly discernible, particularly the strong beats

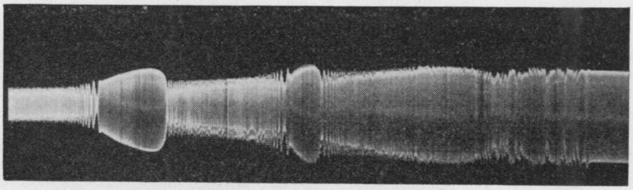

Fig. 8—Oscillogram showing character of oscillations for $C_0=0.7$, $\Delta C=0.2$ microfarad, and $h=0.13$.

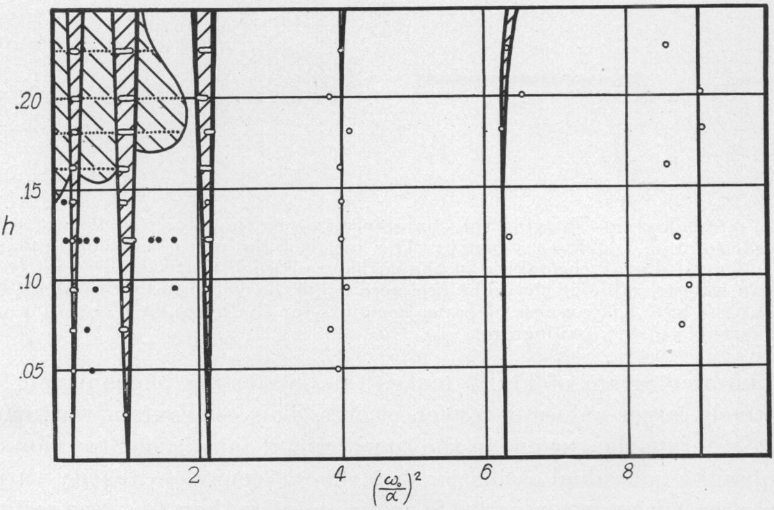

Fig. 9—Plot of the experimentally determined points, showing the character of the oscillations as a function of the several circuit parameters. Legend: white circles = substantially sinusoidal oscillations, black circles = periodic nonsinusoidal oscillations, small dots = no oscillations of any kind. A comparison should be made with the theoretically obtained curve of Fig. 1.

and large amplitude sinusoidal oscillations occurring for $\alpha \cong 2\omega_0$ and $\alpha \cong \omega_0$, respectively, (the first and second prominent maxima of the oscillogram, counted from the left).

A graphical representation of the results of this study may be made in a form similar to Fig. 1. Such a plot is reproduced in Fig. 9. It allows a direct comparison between theory and experiment. It is seen at once that the general agreement is quite good despite the approximations involved in the analysis and pointed out in the first section of this paper. The unstable oscillations of the theoretical case represented in Fig. 1 occur in the experimental curves of Fig. 9 for the same values of h and $(\omega_0/\alpha)^2$, but in the latter figure "unstable" is to be interpreted as an oscillation having practically sinusoidal form and a relatively large amplitude. For values of the circuit parameters bordering on an unstable adjustment large beats take place, while for values remote from an unstable region the wave form is quite complex. Other oscillations, periodic in character but of a nonsinusoidal wave form, may be located in the region of stable adjustment for certain parameter values. A third type of region is to be found in the experimental data, although absent in the theoretical one, in which absolutely no oscillations occur. This latter divergence from the theoretical predictions is the only prominent one found, and is the result of factors inherent in the actual physical circuit but not considered in the theoretical analysis.

Conclusions

The agreement between theory and experiment is quite good, despite the idealized circuit upon which this theory is based. Periodic oscillations of frequency α or $\alpha/2$ having a practically sinusoidal form and a relatively large amplitude occur when $\alpha \cong (2\omega_0/n)$, $n=1, 2, 3, 4, \cdots$, although this phenomenon is most pronounced when $n=1$ or 2. These sinusoidal oscillations correspond to an unstable condition of the circuit, for which the current amplitude would rise beyond all bounds were physical limitations not imposed by the apparatus, as energy is then fed continually into the electrical system from the mechanical one (i.e. the mechanical energy expended in varying the condenser is converted into electrical energy in the circuit). It is only for such an unstable adjustment that the wave form is sinusoidal. Oscillations of the stable type have, in general, a quite complex character which varies with the circuit parameters; they may be conveniently classified into three categories, namely; (1) periodic oscillations of nonsinusoidal form and of fundamental frequency different from α or $\alpha/2$; (2) oscillations of a beat type in which the amplitude periodically waxes and wanes; and (3) oscillations having no definite periodic properties and appearing as a continually transient phenomenon (periodicity may occur in this type, but the period is so long that the practical consequences are as though no periodicity existed).

The complete failure of oscillations for several ranges of parameter values was a phenomenon not directly explainable by the theory. It is believed that the explanation lies in the fact that the capacity variation was secured by a switching method with its attendant loss of energy already discussed. The physical conditions were not all included in the differential equation, and consequently the solution would not necessarily contemplate every experimental observation; such is the case with this complete absence of oscillations for certain adjustments.

A circuit in which the inductance is periodically varied should behave in precisely the same manner as the one with periodically varying capacitance studied here. This follows from the symmetry in L and C of the differential equation (1).

It may be mentioned that the phenomena discussed in this paper represent what may ordinarily be thought of as a very exaggerated case of frequency modulation in which the degree of modulation and the modulation frequency are both of magnitude comparable with the middle frequency. Some of the observed effects are met with in the acoustical application[8] of frequency modulation, viz., the warble tone. They are not thought to occur in any radio-frequency case of present occurrence (this would mean a modulating frequency comparable to or larger than the carrier frequency) but could be produced by suitable apparatus if desired.

[8] W. L. Barrow, Proc. I.R.E., vol. 20, p. 1635; October, (1932).